Existence and Flow Invariance of Solutions to Non-autonomous Partial Differential Delay Equations

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Dedicated to my husband and my son Ali

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Contents

Preface

We study existence and flow invariance of (mild) solutions to non-autonomous partial differential delay equations of the general form

$$\text{(FDE)} \quad \begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t), & 0 \le s \le t \\ u_s = \varphi. \end{cases}$$

Here $B(t) \subset X \times X$ is a time dependent (possibly multivalued) nonlinear operator in a Banach (state) space X with – possibly – time dependent domain $D(B(t))$. For $I = \mathbb{R}^-$ or $I = [-R, 0]$, $R > 0$, $u_t : I \to X$ is the history of u up to t defined by $u_t(\xi) = u(t + \xi)$, $\xi \in I$, and $\varphi : I \to X$ is a given initial history out of a space E of functions from I to X. Moreover, $F(t, .)$ is a given history-responsive operator which is Lipschitz continuous on a (possibly "thin") subset $\hat{E}(t)$ of E.
Equations of type (FDE) arise in the modeling of evolutionary processes governed by – possibly multivalued 'partial differential expressions ' for which the time rate of change at time t depends in an essential way on the history of the process up to time t, or at least on some finite part of it. Typical examples of such arise from the investigation of materials with thermal– or shape–memory, of biochemical reactions, and of population models.
This problem has been considered in quite a number of papers. In these papers mostly the operators $B(t)$ are assumed to be $\alpha(t)$-m-accretive, $\alpha(t) \in \mathbb{R}$, with $cl(D(B(t)))$ independent of t and the operators $F(t, .)$ are supposed to be globally Lipschitz continuous, or at least globally defined and Lipschitz on bounded sets. Also in the search for "classical" solutions, there are usually special assumptions on the geometry of the state space X. (Compare the references in Chapter 2).

For applications, these can be severe restrictions. For concrete models in population dynamics, for instance, the natural state space X is an L^1–space, and the history–responsive operators $F(t, .)$ may not be globally defined, much less globally Lipschitz, but may only be defined on "thin" subsets $\hat{E}(t)$ of the initial history space. (For a class of concrete models, see Sections 2.5 and 3.4).

An appropriate solution theory thus requires not only existence of solutions, but also their flow-invariance. Also, for the general case of nonlinear operators $B(t)$ and geometrically "bad" Banach spaces, the search for classical solutions has to be extended to the search for mild solutions, as usual for nonlinear evolution problems.

In this thesis, we shall present two approaches to flow invariance of solutions to (FDE): (a) under range conditions on $(B(t))_{t\geq 0}$;
(b) under subtangential conditions.
The general setup is as follows:
(G1) $(B(t))_{t\geq 0}$ is a family of operators in a Banach space (with possibly time–varying domain $D(B(t))$), such that for $[x_i, y_i] \in B(t_i)$, $i \in \{1,2\}$, $t_2 \leq t_1$, and $0 < \lambda < \lambda_0$;

$$(1-\lambda\alpha)\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| + \lambda\|f(t_1) - f(t_2)\|L_1(\|x_2\|),$$

where $\alpha \in \mathbb{R}$, $\lambda_0 > 0$ is such that $\lambda_0\alpha < 1$, $f : \mathbb{R}^+ \to X$ is a continuous function, and $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ a nondecreasing bounded function (on bounded sets).

(G2) $\hat{X}(t)$ and $\hat{E}(t)$, $t \geq 0$ are closed subsets of X, respectively, of E such that for $x \in \hat{X}(t+\lambda)$, $\psi \in \hat{E}(t)$, and $\lambda > 0$ small enough, if $\varphi^{\psi}_{x,\lambda} \in E$ is the solution to $\varphi - \lambda\varphi' = \psi$, $\varphi(0) = x$, then $\varphi^{\psi}_{x,\lambda} \in \hat{E}(t+\lambda)$.

(G3) $F : \bigcup\limits_{t\geq 0}\{t\}\times\hat{E}(t) \to X$ is continuous and bounded on bounded sets.

(a) In Chapter 2, we shall extend the approach of [58] to the fully non-autonomous case. More precisely we investigate the problem of existence and flow invariance of mild solutions to the non-autonomous partial differential delay equation $\dot{u}(t) + B(t)u(t) \ni F(t, u_t)$, $t \geq s \geq 0$, $u_s = \varphi$, where the family $(B(t))_{t\geq 0}$ satisfies (G1) and the operators $F(t, .)$ satisfy (G3) and have the following property: for $\varphi \in \hat{E}(t_1)$ and $\psi \in \hat{E}(t_2)$, $0 \leq t_2 \leq t_1$, such that $\|\varphi - \psi\| = \|\varphi(0) - \psi(0)\|$,

$$\|F(t_1, \varphi) - F(t_2, \psi)\| \leq M\|\varphi - \psi\| + \|g(t_1) - g(t_2)\|L_2(\|\psi\|), \tag{0.1}$$

with $g : \mathbb{R}^+ \to X$ a continuous function and $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$ a bounded (on bounded sets) function. We show that under the above assumptions if for all $x \in \hat{X}(t+\lambda)$, $\psi \in \hat{E}(t)$, and $\lambda > 0$ small enough;

$$\left[\psi(0) + \lambda F(t+\lambda, \varphi^{\psi}_{x,\lambda})\right] \in (I + \lambda B(t+\lambda))(D(B(t+\lambda)) \cap \hat{X}(t+\lambda)),$$

then for φ in a certain subset of $\hat{E}(s)$, there exists a mild solution u_φ to (FDE) such that $(u_\varphi)_t \in \hat{E}(t)$, $t \geq s$.

(b) In Chapter 3, we provide a subtangential condition for existence and flow invariance of solutions to (FDE). (For the autonomous (FDE), see [59]). Namely, we show that under certain assumptions on F and the family $(B(t))_{t\geq 0}$ if for all $\psi \in \hat{E}(t)$

$$\liminf_{\lambda\to 0^+} \frac{1}{\lambda}\, d\big(\psi(0) + \lambda F(t,\psi), (I + \lambda B(t+\lambda))(\hat{X}(t+\lambda) \cap D(B(t+\lambda)))\big) = 0,$$

then for any $\varphi \in \hat{E}(s)$, there exists a mild solution u_φ to (FDE) such that $(u_\varphi)_t \in \hat{E}(t)$ and $u_\varphi(t) \in \hat{X}(t)$, $t \geq s$. Here $\hat{E}(t)$ is a family of closed subsets in $\hat{E}_0(t)$ with $\hat{E}_0(t) = \{\varphi \in E \,|\, \varphi(0) \in cl(\hat{X}(t) \cap D(B(t)))\}$ satisfying (G2). In this case, the additional assumptions on F are as below:

$(F1)$ There exists $M > 0$ such that for $\varphi_i \in \hat{E}_0(t_i)$, $i \in \{1, 2\}$, $0 \leq t_2 \leq t_1$, with $\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\|$,

$$\begin{aligned} < F(t_1, \varphi_1) - F(t_2, \varphi_2), \varphi_1(0) - \varphi_2(0) >_+ \;&\leq\; M\|\varphi_1 - \varphi_2\| \\ &+\; \|g(t_1) - g(t_2)\| L_2(\|\varphi_2\|), \end{aligned}$$

where $g : \mathbb{R}^+ \to X$ is a continuous function and $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded (on bounded sets) function.

$(F2)$ there exists $M' > 0$ such that, if $\varphi_1, \varphi_2 \in \hat{E}_0(t)$, $t \geq 0$ with $\varphi_1(0) = \varphi_2(0)$, then

$$\|F(t, \varphi_1) - F(t, \varphi_2)\| \leq M'\|\varphi_1 - \varphi_2\|.$$

In the subtangential case, we shall also need some t–dependence condition on the family $D(B(t)) \cap \hat{X}(t)$ and $\hat{E}(t)$.

In both cases, we associate with (FDE) a family of nonlinear operators $A(t) : D(A(t)) \subset E \to E$ by

$$\begin{cases} D(A(t)) = \{\varphi \in \hat{E}(t) \mid \varphi' \in E, \varphi(0) \in D(B(t)), \varphi'(0) \in F(t, \varphi) - B(t)\varphi(0)\} \\ A(t)\varphi := -\varphi' \,, \varphi \in D(A(t)), \end{cases} \tag{0.2}$$

(in the subtangential case $D(A(t)) \subset \hat{E}_0(t)$). Then, our analysis will be based on the evolution operator associated to the Cauchy problem $\dot{\Phi}(t) + A(t)\Phi(t) = 0$, $\Phi(s) = \varphi$ in the initial history space E.
We also investigate the asymptotic properties (such as asymptotic stability, compactness of the range of solutions, and asymptotic almost periodicity) of the solutions to (FDE).

This thesis is organized as follows:
In Chapter 1, we recall some basic definitions and known facts about accretive operators, mild solutions, and evolution operators.

In Chapter 2, we first explain the assumptions on the initial history space E and give some typical examples for the initial history spaces. In this chapter, we consider (FDE) under the general assumptions (G1)–(G3), condition (0.1) on F, and the above range condition. Then we prove that for all φ in $cl(D(A(s)))$, there exists a mild solution u_φ such that $(u_\varphi)_t \in \hat{E}(t)$, $t \geq s$. Under some additional assumption we also obtain invariance of $\hat{X}(t)$, i.e. $u_\varphi(t) \in \hat{X}(t)$ for $t \geq s$. In Section 2.6 we shall apply these results to population dynamics models. The existence and flow invariance results for (FDE) in the initial history space of L^1 type are presented in Section 2.7.

In Chapter 3, we consider (FDE) under the general assumptions (G1)–(G3), conditions (F1) and (F2), and the subtangential condition explained above. Then we state the main theorem of this chapter, namely we show that for any $\varphi \in \hat{E}(s)$ there exists a mild solution u_φ such that $(u_\varphi)_t \in \hat{E}(t)$ and $u_\varphi(t) \in \hat{X}(t)$, $t \geq s$. Next, we discuss some special cases which are more useful for the applications to the population models. The examples are treated in Section 3.4.
As we indicated before, the proof of the main theorem is based on translating (FDE) to a Cauchy problem in E. In translating the above subtangential condition on (FDE) to a sufficient condition for the existence and flow invariance of mild solutions for the Cauchy problem in E, we have to work with a separate subtangential condition which in the autonomous case was developed by M. Pierre. Thus, in the first section of this chapter, we shall introduce the separate subtangential condition in the non–autonomous case and show that it is sufficient for the existence and flow invariance of mild solutions to the non–autonomous Cauchy problem.

In Chapter 4, we study the asymptotic properties of solutions to (FDE) obtained in the previous chapters. In Section 4.1, we show that if $\alpha + M < 0$, then for the initial history spaces (a) $E = E_v$ where $s \mapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$ for some $\mu > 0$, and (b) $E = C([-R, 0]; X)$, solutions to (FDE) are exponentially stable. In Section 4.2, we consider (FDE) in the initial history space $E = C([-R, 0]; X)$, and show that if the the family $(B(t))_{t\geq 0}$ generates a compact evolution operator, then the evolution operator generated by the family $(A(t))_{t\geq 0}$, defined by (0.2), is also compact. In Section 4.3 we study the relationship between properties (such as having (weakly) relatively compact range, asymptotic almost periodicity, and Eberlein–weak almost periodicity) of u_φ, the mild solution to (FDE), and the corresponding motion $U_A(., 0)\varphi : \mathbb{R}^+ \to E$. Finally, in Section 4.4 we discuss some cases, for which the solutions to (FDE) have relatively compact range. In particular we show that if for some $r_0 > 0$, $B(r_0)$ is α–m–accretive and the resolvents of $B(r_0)$ are compact, then bounded and uniformly continuous solutions to (FDE) have relatively compact range.

1 Preliminaries

In this chapter, we collect some basic facts which will be used later. Throughout this thesis, X will be a real Banach space. Given a subset D of X, $cl(D)$ will denote its (norm–) closure in X and $d(x, D)$ is the distance from x to the set D.

1.1 Accretive operators

In this section we introduce some basic concepts concerning the theory of accretive operators.

A (possibly multivalued) operator $C : D(C) \to 2^X$, represented by its graph $C \subset X \times X$, is called *accretive* if for each $\lambda > 0$ and each pair $[x_i, y_i] \in C$, $i \in \{1, 2\}$,

$$\|x_1 - x_2\| \leq \|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\|.$$

For $\gamma \in \mathbb{R}$, C is called γ-*accretive* if $C + \gamma I$ is accretive, or equivalently for $\lambda > 0$ with $\lambda\gamma < 1$ and each pair $[x_i, y_i] \in C$, $i \in \{1, 2\}$,

$$(1 - \lambda\gamma)\|x_1 - x_2\| \leq \|(x_1 + \lambda y_1) - (x_2 + \lambda y_2)\|.$$

Let $C \subset X \times X$ be a $\gamma-$ accretive operator, and $\lambda > 0$ with $\lambda\gamma < 1$. Then its resolvents $J_\lambda^C := (I + \lambda C)^{-1} : R(I + \lambda C) \to D(C)$ are single–valued, and moreover

$$\left\|J_\lambda^C x - J_\lambda^C y\right\| \leq \frac{1}{1 - \lambda\gamma}\|x - y\|.$$

The operator $C_\lambda : R(I + \lambda C) \to X$ defined by

$$C_\lambda := \lambda^{-1}(I - J_\lambda^C),$$

is called the *Yosida approximation* of C. By definition, if $x \in R(I + \lambda C)$, then $[J_\lambda^C x, C_\lambda x] \in C$.
C is called γ- m-accretive if C is γ–accretive, and in addition $R(I + \lambda C) = X$ for all $\lambda > 0$, $\lambda\gamma < 1$.

Definition 1.1. The duality map J of X into 2^{X^*} is defined by

$$J(x) = \{x^* \,|\, x^* \in X^*,\ < x^*, x >= \|x\|, \text{ and } \|x^*\| \leq 1\} \quad \text{for all} \quad x \in X. \tag{1.1}$$

We shall also need the following notion:
For each $\lambda > 0$, and $x, y \in X$ set $< y, x >_\lambda = \frac{\|x+\lambda y\| - \|x\|}{\lambda}$. Then for fixed $x, y \in X$, the map $\lambda \mapsto < y, x >_\lambda$ is nondecreasing. Thus for every $x, y \in X$ we can define

$$< y, x >_+ = \lim_{\lambda \to 0^+} < y, x >_\lambda = \inf_{\lambda > 0} < y, x >_\lambda . \tag{1.2}$$

$< y, x >_+ : X \times X \to \mathbb{R}$ is upper–semicontinuous, and $|< y, x >_+| \leq \|y\|$. The following result concerns the relationship between J and $< ., . >_+$.

Proposition 1.2. [3, Proposition 2.13]. *Let $x, y \in X$. Then*

$$< y, x >_+ = \max\{< x^*, y > \,|\, x^* \in J(x)\}.$$

For more properties of this function, see for instance Chapter 2 in [3].

In the next theorem, we characterize accretive operators in several equivalent ways:

Theorem 1.3. [3, Theorem 2.15]. *Let $C \subset X \times X$. Then the following are equivalent:*
(i) C is accretive.
(ii) $(I + \lambda C)^{-1}$ is nonexpansive for $\lambda > 0$.
(iii) $< y - \hat{y}, x - \hat{x} >_+ \geq 0$ whenever $[x, y]$, $[\hat{x}, \hat{y}] \in C$.
(iv) If $[x, y]$, $[\hat{x}, \hat{y}] \in C$, then there exists $x^* \in J(x - \hat{x})$ such that

$$< x^*, y - \hat{y} > \geq 0.$$

The following lemma can be read from [3, Proposition 4.3].

Lemma 1.4. *Let C be a $\gamma-$accretive operator in X, $\lambda > 0$ with $\lambda\gamma < 1$. Then the following hold:*

(i) If $x \in D(J_\lambda^C) \cap D(C)$, then

$$\left\|x - J_\lambda^C x\right\| \leq \frac{\lambda}{1 - \lambda\gamma} \inf\{\|y\| : \quad y \in Cx\}.$$

(ii) If $x \in D(J_\lambda^C)$, then for all $[x_0, y_0] \in C$

$$\left\|J_\lambda^C x - x\right\| \leq \frac{2 - \lambda\gamma}{1 - \lambda\gamma}\|x - x_0\| + \frac{\lambda}{1 - \lambda\gamma}\|y_0\|.$$

1.2 Non–autonomous Cauchy problems

Consider the non–autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) + C(t)u(t) \ni f(t) \;\; 0 \leq s \leq t \\ u(s) = u_0, \end{cases} \tag{1.3}$$

where $C(t) : D(C(t)) \to 2^X$, $t \geq 0$ is a time dependent nonlinear operator acting in X with (possibly) time-dependent domain $D(C(t))$ and $f \in L^1_{loc}(s, \infty; X)$.
Let $T > s$. A continuous function $u : [s, T] \to X$ with $u(s) = u_0$ is called a *strong* solution to (1.3) on $[s, T]$ if u is locally absolutely continuous and differentiable a.e on (s, T) such that $\dot{u}(t) + C(t)u(t) \ni f(t)$ a.e $t \in [s, T]$.
A continuous function $u : [s, T] \to X$ with $u(s) = u_0$ is called a *mild* solution of (1.3) on $[s, T]$ if there exists a sequence of DS-approximate solutions u_n such that u_n converges to u on compact subintervals of $[s, T)$ as $n \to \infty$.
Here, by a DS-approximate solution u_n of (1.3) we mean a step function u_n with

$$u_n(t) = \begin{cases} u_0^n & t = s \\ u_k^n & t \in (t_{k-1}^n, t_k^n], \end{cases} \tag{1.4}$$

where $s = t_0^n < t_1^n < \ldots < t_{N_n}^n \leq T$, and

$$d_n = \max\{\max_{1 \leq k \leq N_n} (t_k^n - t_{k-1}^n), T - t_{N_n}\} \to 0, \quad \text{as} \quad n \to \infty. \tag{1.5}$$

Moreover, the elements u_k^n solve the implicit difference scheme

$$\frac{u_k^n - u_{k-1}^n}{t_{k-1}^n - t_k^n} + C(t_k^n)u_k^n \ni z_k^n, \;\; k \in \{1, \ldots, N_n\}, \tag{1.6}$$

with $u_0^n \to u_0$ as $n \to \infty$, and $z_1^n, \ldots, z_{N_n}^n \in X$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{N_n} \int_{t_{k-1}^n}^{t_k^n} \|f(\tau) - z_k^n\| d\tau = 0. \tag{1.7}$$

Suppose that the family $\{C(t), 0 \leq t \leq T\}$, $C(t) \subset X \times X$, of operators satisfies the following condition:

$(A.1)$ There exists $\gamma \geq 0$, a continuous function $h : (a, b) \to X$, $-\infty \leq a \leq 0 < T < b \leq \infty$, and a bounded (on bounded subsets) function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $[x_1, y_1] \in C(t_1), [x_2, y_2] \in C(t_2), 0 \leq t_2 \leq t_1 \leq T$, and $0 < \lambda < \lambda_0$;

$$(1 - \lambda\gamma)\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| + \lambda\|h(t_1) - h(t_2)\|L(\|x_2\|)$$

where $\lambda_0 > 0$ is such that $\lambda_0\gamma < 1$.
A continuous function $u : [s, T] \to X$ is called an *integral* solution to the Cauchy problem (1.3) on $[s, T]$ if $u(s) = u_0$, and for any $s \le t_2 \le t_1 \le T$, and $[\tilde{x}, \tilde{y}] \in C(r)$, $r \in [s, T]$;

$$\begin{aligned} &\|u(t_1) - \tilde{x}\| - \|u(t_2) - \tilde{x}\| \\ &\le \int_{t_2}^{t_1} \big[\gamma\|u(\tau) - \tilde{x}\| + < f(\tau) - \tilde{y}, u(\tau) - \tilde{x} >_+ + C\|h(\tau) - h(r)\|\big] d\tau, \end{aligned} \tag{1.8}$$

where $C = \max\{L(\|\tilde{x}\|), L\big(\sup_{s\le\tau\le T} \|u(\tau)\|\big)\}$.

Remark 1.5. The inequality (1.8) is equivalent to

$$\begin{aligned} &e^{-\gamma t_1}\|u(t_1) - \tilde{x}\| - e^{-\gamma t_2}\|u(t_2) - \tilde{x}\| \le \\ &\int_{t_2}^{t_1} e^{-\gamma\tau} < f(\tau) - \tilde{y}, u(\tau) - \tilde{x} >_+ d\tau + C\int_{t_2}^{t_1} e^{-\gamma\tau}\|h(\tau) - h(r)\| d\tau, \end{aligned} \tag{1.9}$$

Definition 1.6. A continuous function $u : (I + s) \cup [s, T] \to X$ is called a *strong*, *mild*, or *integral* solution of (FDE) if $u_s = \varphi$ and, on $[s, T]$, u is a solution of the respective kind to the Cauchy problem

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni f(t), & s \le t \le T \\ u(s) = \varphi(0), \end{cases} \tag{1.10}$$

with $f(t) = F(t, u_t)$.

1.3 Evolution operators

Given $T > 0$, let $\{D(s) \,|\, 0 \le s \le T\}$ be a family of subsets of X. Set $\Delta = \{(t, s) \,|\, 0 \le s \le t \le T\}$. A family $U = \{U(t, s) \,|\, (t, s) \in \Delta\}$ of operators (or in short $U(t, s)$), $U(t, s) : D(s) \to D(t)$ is called an evolution operator if it has the following properties:
(i) $U(s, s)x = x$, and $U(t, r)U(r, s)x = U(t, s)x$ for all $x \in D(s)$ and, $0 \le s \le r \le t \le T$,
(ii) For each $s \in [0, T]$ and $x \in D(s)$, the function $t \to U(t, s)x$ is continuous on $[s, T]$.
We shall call the evolution operator $U(t, s)$, $(t, s) \in \Delta$ of type γ if

$$\|U(t, s)x - U(t, s)y\| \le e^{\gamma(t-s)}\|x - y\| \quad \text{for all} \quad x, y \in D(s), \text{ and } \quad (t, s) \in \Delta.$$

If U is an evolution operator of type γ it has the following property ;
(iii) If $0 \le s_n \le t_n \le T$ with $s_n \downarrow s\,, t_n \to t$, and $x_n \in D(s_n)$ such that $x_n \to x$, and $x \in D(s)$, then $U(t_n, s_n)x_n \to U(t,s)x$.

The next theorem states an existence result concerning Cauchy problem

$$\text{(CP)} \quad \begin{cases} \dot{u}(t) + C(t)u(t) \ni 0, & t \ge s \\ u(s) = u_0. \end{cases}$$

Theorem 1.7. [47, Theorem 3.6]. *Given $T > 0$. Suppose that the family $\{C(t), 0 \le t \le T\}$, $C(t) \subset X \times X$, of operators satisfies (A.1) and the following further conditions:*

(A.2) For $0 \le t \le T$ and for all $x \in cl(D(C(t)))$,

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} d(x, R(I + \lambda C(t+\lambda))) = 0.$$

(A.3) The domain $D(C(t))$ depends on $t \in (s,T]$ in the following way:
If $t_n \uparrow t \in (s,T]$, $x_n \in D(C(t_n))$, and $x_n \to x \in X$, then $x \in cl(D(C(t)))$.

Then we have :
(i) For every $s \in [0,T)$ and $u_0 \in cl(D(C(s)))$ the problem (CP) admits a sequence u_n of DS-approximate solutions, which is uniformly convergent on compact subintervals of $[s,T)$ to a continuous function $u = u(;s,u_0)$ with $u(t) \in cl(D(C(t)))$. Moreover every sequence of DS-approximate solutions is uniformly convergent to the same u which is the unique integral solution of (CP).
(ii) The family $\{V(t,s) \,|\, V(t,s) : cl(D(C(s))) \to cl(D(C(t)))\}$ defined by;

$$V(t,s)u_0 = u(t), u_0 \in cl(D(C(s))), s \le t \le T, \tag{1.11}$$

is an evolution operator of type γ.
(iii) If u is a strong solution to (CP), then $u(t) = V(t,s)u_0$.
(iv) If X is reflexive, h is of bounded variation and $u_0 \in D(C(s))$, then $u(t) = V(t,s)u_0$ is the unique strong solution to (CP).

Remark 1.8. Theorem 1.7 actually holds for any $\gamma \in \mathbb{R}$; and Theorem 1.7(iv) holds, if X just has the Radon–Nikodym property.

In what follows we shall call the family $\{V(t,s) \mid \; s \le t \le T\}$ as defined in (1.11) the evolution operator generated by $C(t)$.

Definition 1.9. The evolution operator $U(t,s)$ is said to be compact if for every t,s with $0 \leq s < t$, $U(t,s)$ maps bounded subsets of $D(s)$ to relatively compact subsets of $D(t)$.

For $t = s$, $U(s,s)$ is the identity operator on $D(s) \subset X$ which is not compact if X is of infinite dimension. Moreover, if $U(t_0,s)$ is compact for some $t_0 > s$, then $U(t,s)$ is also compact for all $t > t_0$ (this is because $U(t,s) = U(t,t_0)U(t_0,s)$).

Definition 1.10. The evolution operator $U(t,s)$ is said to be equicontinuous if for every $0 \leq s < t$, and each bounded set $K \subset D(s)$, the family of functions $\{U(.,s)x \mid x \in K\}$ is equicontinuous at t. i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|U(t',s)x - U(t,s)x\| < \varepsilon \quad \text{for all} \quad t \in (t'-\delta, t'+\delta), \text{and} \quad x \in K.$$

If $U(t,s)$, is a compact evolution operator, then for each $0 \leq s < t$, and bounded set $K \subset D(s)$, $U(.,s)K$ is equicontinuous at t.
The following result is the extension of Brezis' theorem. See [47, Theorem 5.1]

Theorem 1.11. *Let $(U(t,s))_{0\leq s\leq t\leq T}$ be the evolution operator generated by $C(t)$ via* Theorem 1.7. *Suppose also that each $C(t)$ is m–accretive. Then $U(t,s)$ is compact if and only if $U(t,s)$ is equicontinuous, and for any $r \in [0,T]$, and $\lambda > 0$, $J_\lambda^{C(r)}$ is compact.*

For all these notions and the general theory of accretive operators and evolution equations, the reader is referred to [3, 9, 19, 37, 44, 47].

2 Existence and flow-invariance under local range conditions

[1]In this chapter we study existence and flow invariance of (mild) solutions to non-autonomous partial differential delay equations of the general form

$$\text{(FDE)} \quad \begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t), & 0 \le s \le t \\ u_s = \varphi. \end{cases}$$

Here $B(t) \subset X \times X$ is a time dependent (possibly multivalued) nonlinear operator in a Banach (state) space X with – possibly – time dependent domain $D(B(t))$. For $I = \mathbb{R}^-$ or $I = [-R, 0]$, $R > 0$, $u_t : I \to X$ is the history of u up to t defined by $u_t(\xi) = u(t + \xi)$, $\xi \in I$, and $\varphi : I \to X$ is a given initial history out of a space E of functions from I to X. Moreover, $F(t, .)$ is a given history-responsive operator which is Lipschitz continuous on a (possibly "thin") subset $\hat{E}(t)$ of E with Lipschitz constant $M > 0$.
For the autonomous counterpart of (FDE), in a series of papers Ruess [54, 55, 56, 58] and Ruess/Summers [60, 61, 62] have developed a "local" approach to global existence and flow invariance, and most importantly, F only defined on prescribed subsets $\hat{E}$ of E. In this chapter, we shall extend the approach of [58] to the fully non-autonomous case. This approach roughly consists of the following program:
Given closed subsets $\hat{X}(t) \subseteq X$, and $\hat{E}(t) \subseteq E$, we shall specify conditions on $\hat{X}(t)$, $\hat{E}(t)$, $F(t, .) : \hat{E}(t) \to X$, and the family $B(t)$ of nonlinear operators, $B(t) \subset X \times X$, such that if the operators $A(t)$ in E are defined by

$$\begin{cases} D(A(t)) = \{\varphi \in \hat{E}(t) \mid \varphi' \in E, \varphi(0) \in D(B(t)), \varphi'(0) \in F(t, \varphi) - B(t)\varphi(0)\} \\ A(t)\varphi := -\varphi' \, , \varphi \in D(A(t)), \end{cases}$$

then $A(t)$ generates an evolution operator $\{U(t, s) | \; 0 \le s \le t\}$, $U(t, s) : cl(D(A(s))) \to$

[1]A shortened version of this chapter has been submitted for publication.

$cl(D(A(t)))$, such that, for $\varphi \in cl(D(A(s)))$ the function u_φ, defined by

$$u_\varphi(t) = \begin{cases} \varphi(t-s), & I \ni t-s \le 0 \\ (U(t,s)\varphi)(0), & t \ge s \end{cases}$$

is a mild solution for (FDE), and $(u_\varphi)_t \in \hat{E}(t)$ for all $t \ge s$.

For previous results in this direction, mostly for $B(t)$ α–m–accretive and $F(t,.)$ globally defined, both in the autonomous and the non–autonomous case, the reader is referred to [6, 7, 12, 13, 14, 15, 16, 29, 30, 31, 32, 33, 34, 35, 36, 45, 53, 57, 58, 69, 70, 71, 72, 73, 74], as well as, for a survey, to [56] and the further references therein. For a different kind of local approach in the non–autonomous case, without flow–invariance, but leading to local existence of solutions, we refer to [17, 18].

2.1 The initial history space

Given $I = (-\infty, 0]$ or $I = [-R, 0]$ for some $R > 0$, the initial history space E is assumed to be a Banach space of continuous functions $\varphi : I \to X$ with the following properties:

(E.1) (a) For all $\varphi \in E$, $\|\varphi(0)\| \le \|\varphi\|$.
(b) For all $x \in X$, $\bar{x} \in E$, where $\bar{x}(s) = x, s \in I$.
(c) For φ, (φ_n) in E, if $\|\varphi_n - \varphi\| \to 0$, then for all $s \in I$, $\|\varphi_n(s) - \varphi(s)\| \to 0$.

(E.2) If $\lambda > 0, x \in X, \psi \in E$ and $\varphi_x \in C^1(I; X)$ is the solution to $\varphi_x - \lambda\varphi_x' = \psi$, $\varphi_x(0) = x$, then $\varphi_x \in E$ and $\|\varphi_x\| \le \max\{\|x\|, \|\psi\|\}$.

(E.3) (a) If $x : I \cup [0, \infty) \to X$ is continuous and $x_{|_I} \in E$, then $x_t \in E$ for each $t \ge 0$ and the map $t \to x_t$ is continuous from $\mathbb{R}^+$ into E.
(b) There exist $M_0 \ge 1$, and a locally bounded function $M_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, given $x : I \cup \mathbb{R}^+ \to X$ as in (a) above,

$$\|x_t\| \le M_0\|x_0\| + M_1(t) \sup\{\|x(s)\| : 0 \le s \le t\} \quad \text{for all} \quad t \ge 0.$$

Remark 2.1. For the finite delay case, usually $E = C([-R, 0]; X)$ with supremum norm. For the infinite delay case, the following spaces are common examples with (E.1)-(E.3): E can be taken as a weighted sup–norm space of the type $E_v = \{\varphi \in C(\mathbb{R}^-, X) \mid v\varphi \in BUC(\mathbb{R}^-, X)\}$, with norm $\|\varphi\|_v := \sup\{v(s)\|\varphi(s)\| : s \in \mathbb{R}^-\}$, where the (weight-) function $v : \mathbb{R}^- \to (0, 1]$ has the following properties:

(v1) v is continuous, nondecreasing, and $v(0) = 1$;
(v2) there exists a constant $M_v \geq 0$ such that

$$\lim_{s \to 0^-} \frac{v(s+u)}{v(s)} = 1 \quad \text{uniformly over} \quad u \in I.$$

Typical such weight functions are $v(s) \equiv 1$ (with, in this case, $E_v = BUC(\mathbb{R}^-, X)$ with sup-norm), $v(s) = e^{\mu s}$, or $v(s) = (1 + |s|\ )^{-\mu}$, $\mu > 0$ (spaces of 'fading memory type'), c.f. [60]. The Banach spaces E_v are sometimes called UC_g- spaces, $v = 1/g$, and have been considered by various authors. Aside from E_v, also the following subspaces fulfill axioms (E.1)-(E.3):
$E_{v_l} = \{\varphi \in E_v \mid \lim_{s\to-\infty} v(s)\varphi(s) \text{ exists}\}$, and
$E_{v_0} = \{\varphi \in E_v \mid \lim_{s\to-\infty} v(s)\varphi(s) = 0\}$, in case $\lim_{s\to-\infty} v(s) = 0$.
Here, as well as for axiom (E.3) above, we refer to [25, 28] and the further references listed therein.

We will adopt the following notations:
1. $E_0 = \{\varphi \in E| \ \ \varphi(0) = 0\}$;
2. for $\lambda > 0$, the function $e_\lambda : I \to \mathbb{R}^+$ is defined by $e_\lambda(s) = \exp(s/\lambda), s \in I$.

2.2 Assumptions

For a fixed $T > 0$, we consider the following assumptions:

(B.1) $(B(t))_{0\leq t\leq T}$ is a family of operators $B(t) \subset X \times X$ such that there exist $\alpha \in \mathbb{R}$, a continuous function $f : [0, T] \to X$, and a nondecreasing bounded function (on bounded sets) $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $[x_i, y_i] \in B(t_i), i \in \{1, 2\}$, and $0 \leq t_2 \leq t_1 \leq T$,

$$(1 - \lambda\alpha)\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| + \lambda\|f(t_1) - f(t_2)\|L_1(\|x_2\|), \tag{2.1}$$

for all $\lambda > 0$ with $\lambda\alpha < 1$.

(B.2) $\hat{E}(t)$, $t \in [0, T]$, are closed subsets of E such that
(*i*) $F : \bigcup\limits_{t\in[0,T]} \{t\}\times\hat{E}(t) \to X$ is continuous and maps bounded sets to bounded sets.
(*ii*) There exists $M > 0$ such that for $\varphi \in \hat{E}(t_1)$ and $\psi \in \hat{E}(t_2)$, $0 \leq t_2 \leq t_1 \leq T$, with $\|\varphi - \psi\| = \|\varphi(0) - \psi(0)\|$,

$$\|F(t_1, \varphi) - F(t_2, \psi)\| \leq M\|\varphi - \psi\| + \|g(t_1) - g(t_2)\|L_2(\|\psi\|) \tag{2.2}$$

where $g : [0,T] \to X$ is a continuous function and $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded (on bounded sets) function.

(B.3) $\hat{X}(t)$, $t \in [0,T]$, are closed subsets in X such that for $x \in \hat{X}(t+\lambda)$, $\psi \in \hat{E}(t)$, and $\lambda > 0$ with $\lambda\omega < 1$, $\omega := \max\{0, M+\alpha\}$,
(i) if $\varphi^{\psi}_{x,\lambda} \in E$ is the solution to $\varphi - \lambda\varphi' = \psi$, $\varphi(0) = x$, then $\varphi^{\psi}_{x,\lambda} \in \hat{E}(t+\lambda)$, and moreover,

(ii) $\left[\psi(0) + \lambda F(t+\lambda, \varphi^{\psi}_{x,\lambda})\right] \in (I + \lambda B(t+\lambda))(D(B(t+\lambda)) \cap \hat{X}(t+\lambda))$.

Our subsequent results will be based on these assumptions.

2.3 Existence of mild solutions to (FDE)

In this section, we present the main result of this chapter on existence and flow-invariance of mild solutions to

$$\text{(FDE)} \quad \begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t), & 0 \le s \le t \\ u_s = \varphi. \end{cases}$$

We will base our techniques of proof of the existence of mild solutions on the usual approach via a Cauchy problem in E, which in this local case follows the (autonomous) approach of [58]. We define a family of nonlinear operators $A(t) : D(A(t)) \subset E \to E$ by

$$\begin{cases} D(A(t)) = \{\varphi \in \hat{E}(t) \mid \varphi' \in E, \varphi(0) \in D(B(t)), \varphi'(0) \in F(t,\varphi) - B(t)\varphi(0)\} \\ A(t)\varphi := -\varphi' \,, \varphi \in D(A(t)). \end{cases} \tag{2.3}$$

Theorem 2.2. *If $(B.1)$, $(B.2)$ and $(B.3)$ hold, then given $s \in [0,T)$ and $\varphi \in cl(D(A(s)))$, (FDE) has a mild solution u_φ, such that $(u_\varphi)_t \in \hat{E}(t)$ for all $t \in [s,T]$.*

Proof. Our proof will be divided up into the following three steps. (Compare the autonomous case [58].)

Step 1. The family $A(t)$ generates an evolution operator $\{U(t,s) \mid 0 \le s \le t\}$, $U(t,s) : cl(D(A(s))) \to cl(D(A(t)))$ such that $\|U(t,s)\varphi_1 - U(t,s)\varphi_2\| \le e^{\omega(t-s)}\|\varphi_1 - \varphi_2\|$, where $\omega = \max\{0, M+\alpha\}$.

Step 2. For $\varphi \in cl(D(A(s))$, if we define $u_\varphi : (I+s) \cup [s,T] \to X$ by

$$u_\varphi(t) = \begin{cases} \varphi(t-s), & I \ni t-s \le 0 \\ (U(t,s)\varphi)(0), & t \ge s \end{cases}$$

where $U(t,s)$ is the evolution operator generated in Step 1, then

$$U(t,s)\varphi = (u_\varphi)_t, \qquad s \le t \le T.$$

Step 3. The function u_φ defined in Step 2 is a mild solution to (FDE).

Proof of Step 1. In this part of proof we shall translate our assumptions on $B(t)$ to some conditions on the family $A(t)$, which is sufficient to generate an evolution operator. Namely we prove:

Proposition 2.3. *Under the conditions of* Theorem 2.2, *the Cauchy problem*

$$\begin{cases} \dot\varphi(t) + A(t)\varphi(t) = 0, & s \le t \le T, \\ \varphi(s) = \varphi \in cl(D(A(s))) \end{cases} \tag{2.4}$$

in E admits a unique mild solution.

Proof. Thanks to Theorem 1.7, it is enough to show that $(A.1)$, $(A.2)$ and $(A.3)$ are fulfilled in our setting.

1. Let $\omega = \max\{0, M+\alpha\}$. Given $\lambda > 0$ with $\lambda\omega < 1$. Let $\varphi_1 \in D(A(t_1))$ and $\varphi_2 \in D(A(t_2))$, $s \le t_2 \le t_1 \le T$. As $\psi = \varphi_1 - \varphi_2$ solves the equation $\psi - \lambda\psi' = (\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')$, with $\psi(0) = \varphi_1(0) - \varphi_2(0)$, by (E.2) we have,

$$\|\varphi_1 - \varphi_2\| \le \max\{\|\varphi_1(0) - \varphi_2(0)\|, \|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\|\}.$$

In case $\|\varphi_1(0) - \varphi_2(0)\| \le \|(\varphi_1 - \lambda\varphi_1') - (\varphi_2 - \lambda\varphi_2')\|$, we have the desired inequality. Otherwise, (E.1)(a), implies that $\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\|$. We note that $[\varphi_i(0), F(t_i,\varphi_i) - \varphi_i'(0)] \in B(t_i)$,$i \in \{1,2\}$. Thus using (2.1) and (2.2) we obtain

$$\begin{aligned} (1-\lambda\alpha)\|\varphi_1 - \varphi_2\| &\le \|\varphi_1(0) - \varphi_2(0) + \lambda(-\varphi_1'(0) + \varphi_2'(0))\| \\ &+ \lambda M\|\varphi_1 - \varphi_2\| + \lambda\|g(t_1) - g(t_2)\|L_2(\|\varphi_2\|) \\ &+ \lambda\|f(t_1) - f(t_2)\|L_1(\|\varphi_2(0)\|). \end{aligned}$$

Now since L_1 is nondecreasing, (E.1)(a) implies that

$$\begin{aligned} (1-\lambda(M+\alpha))\|\varphi_1 - \varphi_2\| &\le \|(\varphi_1 - \varphi_2 + \lambda(-\varphi_1' + \varphi_2')\| \\ &+ \lambda\big(\|f(t_1) - f(t_2)\| + \|g(t_1) - g(t_2)\|\big)L(\|\varphi_2\|), \end{aligned} \tag{2.5}$$

where $L = L_1 + L_2$.

Comparing (2.5) to (A.1), we have two control functions f and g. However the proof of Theorem 1.7 remains true. Indeed, it is only the modulus of continuity of the control function which has a role in the proof. For the sake of simplicity, we shall define

$$H(t_1,t_2) := \|f(t_1)-f(t_2)\| + \|g(t_1)-g(t_2)\|. \tag{2.6}$$

Remark 2.4. Since for $\varphi \in D(J_\lambda^{A(t_1)})$, $[J_\lambda^{A(t_1)}\varphi, A_\lambda(t_1)\varphi] \in A(t_1)$, the inequality (2.5) implies that for all $\psi \in D(A(t_2))$, with $t_2 \le t_1$

$$(1-\lambda\omega)\left\|J_\lambda^{A(t_1)}\varphi - \psi\right\| \;\le\; \|\varphi-\psi\| + \lambda\|\psi'\| + \lambda H(t_1,t_2)L(\|\psi\|). \tag{2.7}$$

2. We now show that for $\lambda > 0$ with $\lambda\omega < 1$ (independent of $t \in [s,T]$),

$$cl(D(A(t))) \subseteq \hat{E}(t) \subseteq R(I+\lambda A(t+\lambda)), \tag{2.8}$$

which clearly implies $(A.2)$. Let $\lambda > 0$ with $\lambda\omega < 1$ and $\psi \in \hat{E}(t)$. By (B.3)(ii) we may define

$$T : \hat{X}(t+\lambda) \to \hat{X}(t+\lambda) \quad \text{by} \quad T(x) = J_\lambda^{B(t+\lambda)}\big(\psi(0) + \lambda F(t+\lambda, \varphi_{x,\lambda}^{\psi})\big),$$

with $\varphi_{x,\lambda}^{\psi}$ as defined in (B.3). If $x, y \in \hat{X}(t+\lambda)$, then by (2.2)

$$\begin{aligned}\|Tx - Ty\| \;&\le\; \frac{\lambda}{1-\lambda\alpha}\left\|F(t+\lambda,\varphi_{x,\lambda}^{\psi}) - F(t+\lambda,\varphi_{y,\lambda}^{\psi})\right\| \\ &\le\; \frac{\lambda M}{1-\lambda\alpha}\left\|\varphi_{x,\lambda}^{\psi} - \varphi_{y,\lambda}^{\psi}\right\| \le \frac{\lambda M}{1-\lambda\alpha}\|x-y\|.\end{aligned}$$

Here we have used that $\varphi_{x,\lambda}^{\psi} - \varphi_{y,\lambda}^{\psi}$ is the solution of $\rho - \lambda\rho' = 0$, $\rho(0) = x - y$. Thus, by our choice of ω and λ, T is a strict contraction from $\hat{X}(t+\lambda)$ to $\hat{X}(t+\lambda)$. Hence there exists a unique $z \in \hat{X}(t+\lambda)$ such that,

$$z = T(z) = J_\lambda^{B(t+\lambda)}(\psi(0) + \lambda F(t+\lambda, \varphi_{z,\lambda}^{\psi})).$$

In particular $\varphi_{z,\lambda}^{\psi}(0) = z \in D(B(t+\lambda))$, and

$$\varphi'^{\psi}_{z,\lambda}(0) = \frac{\varphi_{z,\lambda}^{\psi}(0) - \psi(0)}{\lambda} \in F(t+\lambda,\varphi_{z,\lambda}^{\psi}) - B(t+\lambda)\varphi_{z,\lambda}^{\psi}(0)).$$

Therefore, $\varphi_{z,\lambda}^{\psi} \in D(A(t+\lambda))$ and $(I+\lambda A(t+\lambda))\varphi_{z,\lambda}^{\psi} = \psi$.

3. To show (A.3), let $t_n \in [s,T)$, and $\varphi_n \in D(A(t_n))$ such that $t_n \uparrow t \in (s,T]$ and $\varphi_n \to \varphi$ in E as $n \to \infty$. Set $\lambda_n = t - t_n$. By (2.8), we may define $\psi_n = J_{\lambda_n}^{A(t)}\varphi_n$ for n large enough. Then using (2.7) we have

$$\begin{aligned}&(1-\lambda_n\omega)\|\psi_n - \varphi\| \le (1-\lambda_n\omega)\|\varphi_m - \varphi\| + (1-\lambda_n\omega)\|\psi_n - \varphi_m\| \le \\ &\quad (1-\lambda_n\omega)\|\varphi_m - \varphi\| + \|\varphi_n - \varphi_m\| + \lambda_n\|\varphi_m'\| + \lambda_n L(\|\varphi_m\|)H(t,t_m)\end{aligned}$$

for $m \geq 1$. Therefore,

$$\limsup_{n\to\infty} \|\psi_n - \varphi\| \leq 2\|\varphi_m - \varphi\|$$

for $m \geq 1$, which shows that $\psi_n \to \varphi$ as $n \to \infty$. Since $\psi_n \in D(A(t))$ for $n \geq 1$, it follows that $\varphi \in cl(D(A(t)))$.
We shall call the evolution operator generated in proposition (2.3) by $U(t,s)$. Theorem 1.7 then implies

$$\|U(t,s)\varphi - U(t,s)\psi\| \leq e^{\omega(t-s)}\|\varphi - \psi\|, \qquad \varphi, \psi \in cl(D(A(s))). \tag{2.9}$$

Proof of Step 2. The following lemma will be needed;

Lemma 2.5. [60, Lemma 2.2]. *Let E satisfy (E-1)-(E-3), then we have:*
(*a*) *For $t \geq 0$, define $S_0(t) : E_0 \to E_0$ by*

$$(S_0(t)\varphi)(s) = \begin{cases} 0 & -t \leq s \leq 0 \\ \varphi(t+s) & s \leq -t, \end{cases}$$

$s \in I$. Then $(S_0(t))_{t\geq 0}$ is a (linear) C_0-semigroup of contractions on E_0 generated by $-A_0$, where

$$D(A_0) = \{\varphi \in E_0 \,|\, \varphi' \in E_0\}, \quad A_0\varphi = -\varphi'.$$

Let $J_{0,\lambda}$ be the resolvent of A_0, then for all $\varphi \in E_0$, and $\lambda > 0$

$$(J_{0,\lambda}\varphi)(\theta) = \frac{e^{\theta/\lambda}}{\lambda}\int_\theta^0 e^{-s/\lambda}\varphi(s)ds, \qquad \theta \in I. \tag{2.10}$$

(*b*) *If A_1 is the operator in E defined by*

$$D(A_1) = \{\varphi \in E_0 \,|\, \varphi' \in E\}, \quad A_1\varphi = -\varphi',$$

then

(b1) *A_1 is accretive, and for all $\lambda > 0$, $R(I + \lambda A_1) = E$. Moreover,*

$$J_{1,\lambda}\varphi = J_{0,\lambda}(\varphi - \varphi(0)) + (1 - e_\lambda)\varphi(0), \quad \textit{for all} \quad \varphi \in E, \tag{2.11}$$

where $J_{1,\lambda}$ is the resolvent of A_1.

(b2) *If $A : D(A) \subseteq E \to E$ is an ω-accretive operator defined by $A\varphi = -\varphi'$, then for all $\lambda > 0$ with $\lambda\omega < 1$;*

$$J^A_\lambda\varphi = J_{1,\lambda}\varphi + e_\lambda(J^A_\lambda\varphi)(0), \qquad \varphi \in R(I + \lambda A). \tag{2.12}$$

To show that $U(t,s)$ acts as a translation, we will follow the idea of proof in [12, Proposition 1]. Let $\varphi \in D(A(s))$ and $t \geq s$. For $\alpha > 0$, define $M_n(\alpha)$ by

$$M_n(\alpha) = \sum_{j=0}^{n-1} \frac{1}{j!}(\alpha n)^j e^{-\alpha n}, \quad n \in \mathbb{N}.$$

It is proved in [20] that $M_n(\alpha) \to 1$ for $\alpha < 1$, and $M_n(\alpha) \to 0$, for $\alpha > 1$.
Since $(\varphi - \varphi(0)) \in E_0$, Lemma 2.5 implies that

$$\lim_{n\to\infty} J^n_{0,t-s/n}(\varphi - \varphi(0)) = S_0(t-s)(\varphi - \varphi(0)), \tag{2.13}$$

for each $t \geq s$.
Now an induction argument on (2.11) as in [20], and (2.13) imply that for all $\theta \in I$, $(J^n_{1,t-s/n}\varphi)(\theta)$ converges to $(S_0(t-s)\varphi)(\theta)$, except possibly at $\theta = -(t-s)$. Set $\lambda_n = \frac{t-s}{n}$, for $n \geq n_0$ such that $\frac{T-s}{n_0}\omega < 1/2$. Using (2.12) and (2.8) we can prove by induction that (compare also [12])

$$\prod_{i=1}^{m} J_{\lambda_n}(s+i\lambda_n)\varphi = J^m_{1,\lambda_n}\varphi + \sum_{i=1}^{m-1} J^i_{1,\lambda_n}(e_{\lambda_n} b_{m-i-1,\lambda_n}(\varphi)), \quad m \in \mathbb{N} \tag{2.14}$$

where $J_\lambda(s+i\lambda) = J_\lambda^{A(s+i\lambda)}$, and $b_{k,\lambda}(\varphi) = (\prod_{i=1}^{k+1} J_\lambda(s+i\lambda)\varphi)(0)$.
It is easy to show that for any $x \in X$ and $\theta \in I$ (compare also [20])

$$\left(J^k_{1,\lambda}(e_\lambda x)\right)(\theta) = e_\lambda(\theta) x \frac{1}{k!}\left(\frac{-\theta}{\lambda}\right)^k. \tag{2.15}$$

We will see that there exists a constant K such that $\|b_{n-i-1,\lambda_n}(\varphi)\| \leq K$ for $i \in \{1,\cdots,n-1\}$ and $n \in \mathbb{N}$. Thus for $\theta \in I$ we have

$$\begin{aligned}
\left\| \left(\prod_{i=1}^{n} J_{\lambda_n}(s+i\lambda_n)\varphi\right)(\theta) - \left(J^n_{1,\lambda_n}\varphi\right)(\theta)\right\| &= \\
\left\| \sum_{i=1}^{n-1} \left(J^i_{1,\lambda_n}(e_{\lambda_n} b_{n-i-1,\lambda_n}(\varphi))\right)(\theta)\right\| &= \\
\left\| \sum_{i=1}^{n-1} e_{\lambda_n}(\theta) b_{n-i-1,\lambda_n}(\varphi) \frac{1}{i!}\left(\frac{-\theta}{\lambda_n}\right)^i\right\| &\leq \\
K\sum_{i=1}^{n-1} \frac{1}{i!}\Big(\frac{-\theta}{\lambda_n}\Big)^i e_{\lambda_n}(\theta) &= K M_n(-\theta/t-s),
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \left\| \left(\prod_{i=1}^{n} J_{\lambda_n}(s+i\lambda_n)\varphi\right)(\theta) - \left(J^n_{1,\lambda_n}\varphi\right)(\theta)\right\| = 0, \quad -\theta > (t-s). \tag{2.16}$$

But $\lim_{n\to\infty}\left(\prod_{i=1}^{n} J_{\lambda_n}(s+i\lambda_n)\varphi\right)(\theta) = (U(t,s)\varphi)(\theta)$. Indeed, set $\varphi_0^n = \varphi$ and $t_0^n = s$, for $n \geq n_0$. Now considering (2.8), we can select $\varphi_1^n \in D(A(t_1^n))$, with $t_1 = s + \frac{t-s}{n}$, such that $\varphi_1^n = J_{\lambda_n}(s+\lambda_n)\varphi_0^n$. Continuing this argument, we can choose sequences $\{t_k^n\}$ satisfying $t_k^n = s + k(\frac{t-s}{n})$ and $\{\varphi_k^n\}$ in $D(A(t_k^n))$ such that

$$\varphi_k^n = \prod_{j=1}^{k} J_{\lambda_n}(s+j\lambda_n)\varphi_0^n \qquad k \in \{1, \cdots, N_n\},$$

where N_n is the largest integer such that $s + N_n(\frac{t-s}{n}) \leq T$. If we define

$$\varphi_n(\bar{t}) = \begin{cases} \varphi_0^n & \bar{t} = s \\ \varphi_k^n & \bar{t} \in (t_{k-1}^n, t_k^n], \end{cases}$$

then $\{\varphi_n\}$ is a sequence of DS-approximate solutions of (3.44). Therefore,

$$\lim_{n\to\infty} \varphi_n(\bar{t}) = U(\bar{t},s)\varphi \quad \text{uniformly over compact subsets of} \quad [s,T).$$

But $t = s + n(\frac{t-s}{n}) = t_n^n$, and therefore

$$U(t,s)\varphi = \lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \prod_{j=1}^{n} (I + \frac{t-s}{n} A(s + j\frac{t-s}{n}))^{-1}\varphi.$$

It now follows from (2.16), that for $t \in [s,T)$,

$$(U(t,s)\varphi)(\theta) = \varphi(t-s+\theta), \qquad I \ni \theta < -(t-s). \tag{2.17}$$

Note that $U(.,s)\varphi$ and φ are both continuous functions on $[s,T]$, and therefore (2.17) holds for all $t \in [s,T]$. Also observe that, since the left– and right–hand sides of (2.17) are each continuous functions of θ on $\theta < -(t-s)$, (2.17) actually holds for $I \ni \theta \leq -(t-s)$.

The continuity of the evolution operator $U(t,s)$ implies that (2.17) is even true for all $\varphi \in cl(D(A(s)))$. We now define,

$$u_\varphi(t) = \begin{cases} \varphi(t-s), & I \ni t-s \leq 0 \\ (U(t,s)\varphi)(0), & t \geq s. \end{cases} \tag{2.18}$$

Then (2.17) implies that,

$$U(t,s)\varphi = (u_\varphi)_t, \quad t \geq s. \tag{2.19}$$

To finish this part of proof, it remains to show that for $\varphi \in D(A(s))$, $b_{k,\lambda_n}(\varphi)$ are bounded for all $k \in \{0, \cdots, n-1\}$, $n \in \mathbb{N}$.

Lemma 2.6. *Assume (2.7) and (2.8) are satisfied. Let $s \in [0,T)$ and $\varphi \in D(A(s))$, then for all $\lambda > 0$, with $\lambda\omega < 1$ and for all $k \in \mathbb{N}_0$ such that $\lambda(k+1) \le T$ we have*

$$\begin{aligned} \left\| \prod_{i=1}^{k+1} J_\lambda(s+i\lambda)\varphi - \varphi \right\| &\le \lambda\|\varphi'\| \sum_{i=1}^{k+1} (1-\lambda\omega)^{-i} \\ &+ \lambda L(\|\varphi\|) \sum_{i=1}^{k+1} (1-\lambda\omega)^{-i} H(s, s+\lambda(k+1-(i-1))). \end{aligned} \tag{2.20}$$

Proof. Let $k=0$. (2.7) yields that

$$\|J_\lambda(s+\lambda)\varphi - \varphi\| \le \lambda\|\varphi'\|(1-\lambda\omega)^{-1} + \lambda L(\|\varphi\|)(1-\lambda\omega)^{-1} H(s, s+\lambda).$$

Thus (2.20) is satisfied for $k=0$.
Assume that (2.20) holds for all $j \le k-1$. Let $\lambda(k+1) \le T$. Then

$$\begin{aligned} \left\| \prod_{i=1}^{k+1} J_\lambda(s+i\lambda)\varphi - \varphi \right\| &= \left\| J_\lambda(s+(k+1)\lambda) \prod_{i=1}^{k} J_\lambda(s+i\lambda)\varphi - \varphi \right\| \\ &\le (1-\lambda\omega)^{-1} \left\| \prod_{i=1}^{k} J_\lambda(s+i\lambda)\varphi - \varphi \right\| + \lambda\|\varphi'\|(1-\lambda\omega)^{-1} \\ &+ \lambda L(\|\varphi\|)(1-\lambda\omega)^{-1} H(s, s+(k+1)\lambda). \end{aligned} \tag{2.21}$$

From (2.21) and the induction assumption we obtain

$$\begin{gathered} \left\| \prod_{i=1}^{k+1} J_\lambda(s+i\lambda)\varphi - \varphi \right\| \le \lambda\|\varphi'\| \sum_{i=1}^{k} (1-\lambda\omega)^{-(i+1)} \quad + \\ \lambda L(\|\varphi\|) \sum_{i=1}^{k} (1-\lambda\omega)^{-(i+1)} H(s, s+\lambda(k-(i-1))) \quad + \\ \lambda\|\varphi'\|(1-\lambda\omega)^{-1} + \lambda L(\|\varphi\|)(1-\lambda\omega)^{-1} H(s, s+(k+1)\lambda), \end{gathered}$$

which clearly is the inequality (2.20).
Since $\|b_{k,\lambda_n}(\varphi) - \varphi(0)\| \le \left\| \prod_{i=1}^{k+1} J_{\lambda_n}(s+i\lambda_n)\varphi - \varphi \right\|$, we conclude from (2.20) that for $n \in \mathbb{N}$ and $k \in \{0, \cdots, n-1\}$,

$$\|b_{k,\lambda_n}(\varphi) - \varphi(0)\| \le \lambda_n(k+1)(1-\lambda_n\omega)^{-n} \left[\|\varphi'\| + 2K_0 L(\|\varphi\|) \right],$$

where $K_0 = \sup\{\|f(\tau)\| \mid \tau \in [0,T]\} + \sup\{\|g(\tau)\| \mid \tau \in [0,T]\}$. Using the estimate

$$(1-\gamma)^{-1} \le e^{2\gamma}, \ 0 \le \gamma \le 1/2,$$

we have $\|b_{k,\lambda_n}(\varphi) - \varphi(0)\| \leq (T-s)e^{2(T-s)\omega}\left[\|\varphi'\| + 2K_0 L(\|\varphi\|)\right]$, and therefore $\|b_{k,\lambda_n}(\varphi)\|$ is bounded .

Proof of Step 3. We finally show that the function u_φ defined by (2.18) is a mild solution to (FDE). To do so it is enough to show that u_φ is a mild solution to the Cauchy problem

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni f(t), & s \leq t \leq T \\ u(s) = \varphi(0), \end{cases} \tag{2.22}$$

with $f(t) = F(t, u_t)$. According to Proposition 2.3 there are sequences (t_k^n), $(\varphi_k^n) \subset D(A(t_k^n))$ and $(\psi_k^n) \subset E$, $k \in \{1, \ldots, N_n\}$ such that $s = t_0^n < t_1^n < \ldots < t_{N_{n-1}}^n < t_{N_n}^n \leq T$, and

$$\frac{\varphi_k^n - \varphi_{k-1}^n}{t_{k-1}^n - t_k^n} + A(t_k^n)\varphi_k^n \ni \psi_k^n \qquad k \in \{1, \ldots, N_n\}, \tag{2.23}$$

with $\varphi_0^n \to \varphi$, $(\varphi_0^n) \subset E$, as $n \to \infty$, and

$$\lim_{n\to\infty} \sum_{k=1}^{N_n} \|\psi_k^n\|(t_{k-1}^n - t_k^n) = 0. \tag{2.24}$$

Moreover if $\varphi_n(t) = \begin{cases} \varphi_0^n & t = s \\ \varphi_k^n & t \in (t_{k-1}^n, t_k^n] \end{cases}$, then

$$\lim_{n\to\infty} \varphi_n(t) = U(t,s)\varphi \quad \text{uniformly over compact subsets of} \quad [s,T). \tag{2.25}$$

Now we show that $u_n(t) = (\varphi_n(t))(0) = \begin{cases} \varphi_0^n(0) & t = s \\ \varphi_k^n(0) & t \in (t_{k-1}^n, t_k^n] \end{cases}$ is a sequence of DS–approximate solutions to (2.22).
Since $\varphi_k^n \in D(A(t_k^n))$, $k \in \{1, \cdots, N_n\}$, (2.23) at zero implies that,

$$\frac{\varphi_k^n(0) - \varphi_{k-1}^n(0)}{t_{k-1}^n - t_k^n} + B(t_k^n)\varphi_k^n(0) \ni \psi_k^n(0) + F(t_k^n, \varphi_k^n). \tag{2.26}$$

We now look at

$$\sum_{k=1}^{N_n} \int_{t_{k-1}^n}^{t_k^n} \|\psi_k^n(0) + F(t_k^n, \varphi_k^n) - F(\tau, U(\tau,s)\varphi)\| d\tau \leq \tag{2.27}$$

$$\sum_{k=1}^{N_n} \|\psi_k^n(0)\|(t_{k-1}^n - t_k^n) + \sum_{k=1}^{N_n} \int_{t_{k-1}^n}^{t_k^n} \|F(t_k^n, \varphi_n(t_k^n)) - F(\tau, U(\tau,s)\varphi)\| d\tau.$$

We note that $D := \{(t, U(t,s)\varphi) \,|\, t \in [s,T]\} \cup \{(t, \varphi_n(t)) \,|\, t \in [s,T],\, n \in \mathbb{N}\}$ is a relatively compact subset of $\bigcup_{t\in[0,T]} \{t\}\times\hat{E}(t)$. Thus F is uniformly continuous on D. This together with (2.24) implies that (2.27) tends to 0 as $n \to \infty$. Now (2.25) implies that $u_n(t)$ converges to $(U(t,s)\varphi)(0)$ uniformly over compact subintervals of $[s,T)$, and this completes the proof.

Remark 2.7. Following the proof of [47, Theorem 3.2 (2)], we can see that u_φ is an integral solution to (FDE). This means that for $r \in [s,T]$ and $[x,y] \in B(r)$, u_φ satisfies the following inequality

$$\begin{aligned} &\|u_\varphi(t_1) - x\| - \|u_\varphi(t_2) - x\| \\ &\leq \int_{t_2}^{t_1} \left[\alpha\|u_\varphi(\tau) - x\| + < F(\tau, u_{\varphi_\tau}) - y, u_\varphi(\tau) - x >_+ + C\|f(\tau) - f(r)\|\right] d\tau \end{aligned} \tag{2.28}$$

for all $s \leq t_2 \leq t_1 \leq T$, where $C = \max\{L(\|x\|), L\big(\sup_{s\leq\tau\leq T} \|u_\varphi(\tau)\|\big)\}$.

Lemma 2.8. *Let the family $C(t)$ of operators satisfy a condition of the form (2.1), and $g_1, g_2 \in L^1(0,T : X)$. If u and v are mild solutions of $\dot{u}(t) + C(t)u(t) \ni g_1(t)$ on $[0,T]$ and $\dot{v}(t) + C(t)v(t) \ni g_2(t)$ on $[0,T]$ respectively, then*

$$\begin{aligned} &e^{-\alpha t_1}\|u(t_1) - v(t_1)\| - e^{-\alpha t_2}\|u(t_2) - v(t_2)\| \leq \\ &\int_{t_2}^{t_1} e^{-\alpha\tau} < g_1(\tau) - g_2(\tau), u(\tau) - v(\tau) >_+ d\tau \leq \int_{t_2}^{t_1} e^{-\alpha\tau}\|g_1(\tau) - g_2(\tau)\| d\tau. \end{aligned} \tag{2.29}$$

Proof. For the case of all $C(t)$ m-accretive, see [47, Theorem 4.1]. For the more general case, combining the method of proofs in [47, Theorem 3.2] and [3, Theorem 6.4] we can still show this result. For the proof under different t-dependence conditions on the family $C(t)$, see [19, Theorem 3] and [16, Proposition 3].

Corollary 2.9. *Let φ, and ψ in $cl(D(A(s)))$, and let u_φ and u_ψ be mild solutions to (FDE) as in Theorem 2.2. Then*

$$\begin{aligned} e^{-\alpha t_1}\|u_\varphi(t_1) - u_\psi(t_1)\| - e^{-\alpha t_2}\|u_\varphi(t_2) - u_\psi(t_2)\| \leq \\ \int_{t_1}^{t_2} e^{-\alpha\tau} < F(\tau, (u_\varphi)_\tau) - F(\tau, (u_\psi)_\tau), u_\varphi(\tau) - u_\psi(\tau) >_+ d\tau \leq \\ \int_{t_2}^{t_1} e^{-\alpha\tau}\|F(\tau, (u_\varphi)_\tau) - F(\tau, (u_\psi)_\tau)\| d\tau. \end{aligned} \tag{2.30}$$

We employ the following notation:

The family $F(.,.)$ is said to be Lipschitz continuous on bounded sets if, given any

S, $r > 0$, there exists $M(S,r) > 0$ such that for all $t \in [0,S]$, and $\varphi, \psi \in \hat{E}(t)$ with $\|\varphi\|, \|\psi\| \le r$,

$$\|F(t,\varphi) - F(t,\psi)\| \le M(S,r)\|\varphi - \psi\|. \tag{2.31}$$

Proposition 2.10. *In the context of Theorem 2.2, if in addition $F(.,.)$ is Lipschitz continuous on bounded sets, then the solutions u_φ to (FDE) with $(u_\varphi)_t \in \hat{E}(t)$ for all $t \ge s$ are unique.*

Proof. Let $\varphi \in \hat{E}(t)$, and u_φ, v_φ be mild solutions to (FDE) with $(u_\varphi)_s = (v_\varphi)_s = \varphi$, $(u_\varphi)_t$, $(v_\varphi)_t \in \hat{E}(t), t \ge s$, and $u_\varphi \ne v_\varphi$. Since $u_\varphi(s) = v_\varphi(s)$, we may define

$$t_0 = \sup\{t \ge s \,|\, u_\varphi = v_\varphi \quad \text{on} \quad [s,t]\}.$$

Then $0 \le t_0 < \infty$, and, by continuity of u_φ, v_φ, $u_\varphi(t_0) = v_\varphi(t_0)$, and $(u_\varphi)_\xi = (v_\varphi)_\xi$ for all $\xi \in [s, t_0]$. Set

$$r = \max\{ \sup_{t_0 \le \xi \le t_0+1} \|(u_\varphi)_\xi\|, \sup_{t_0 \le \xi \le t_0+1} \|(v_\varphi)_\xi\| \},$$

and $K = \sup\{M_1(\xi) \mid 0 \le \xi \le t_0 + 1\}$, with M_1 as in (E.3)(b).
Let $\delta \le \min\{1, \frac{|\alpha|}{2MK(e^{|\alpha|}-1)}\}$, where $M = M(t_0+1, r)$ as in (3.17). (If $\alpha = 0$, let $\delta \le \min\{1, (2MK)^{-1}\}$). We shall assume $\alpha \ne 0$. Choose $t_1 \in (t_0, t_0 + \delta]$ such that

$$\|u_\varphi(\xi) - v_\varphi(\xi)\| \le \|u_\varphi(t_1) - v_\varphi(t_1)\| \quad \text{for all} \quad \xi \in [t_0, t_0 + \delta]. \tag{2.32}$$

From Corollary 2.9 we have

$$\|u_\varphi(t_1) - v_\varphi(t_1)\| \le \int_{t_0}^{t_1} e^{\alpha(t_1-\xi)} \|F(\xi, (u_\varphi)_\xi) - F(\xi, (v_\varphi)_\xi)\| d\xi.$$

Now (E.3)(b) implies that

$$\begin{aligned} \|u_\varphi(t_1) - v_\varphi(t_1)\| &\le \int_{t_0}^{t_1} e^{\alpha(t_1-\xi)} MM_0 \|(u_\varphi)_{t_0} - (v_\varphi)_{t_0}\| d\xi \\ &+ \int_{t_0}^{t_1} e^{\alpha(t_1-\xi)} MK \max_{t_0 \le \tau \le \xi} \|u_\varphi(\tau) - v_\varphi(\tau)\| d\xi \\ &\le MK \|u_\varphi(t_1) - v_\varphi(t_1)\| \int_{t_0}^{t_1} e^{|\alpha|(t_1-\xi)} d\xi. \end{aligned}$$

Therefore

$$\|u_\varphi(t_1) - v_\varphi(t_1)\| \le \delta MK(\frac{e^{|\alpha|} - 1}{|\alpha|}) \|u_\varphi(t_1) - v_\varphi(t_1)\|$$

which contradicts $\|u_\varphi(t_1) - v_\varphi(t_1)\| > 0$.

Remark 2.11. In [7], the following initial history space has been considered: Let $r > 0$ and p a positive function on $(-\infty, 0]$ with the property that $p(0) = 1$, and $p(x)e^x$ is nondecreasing on $(-\infty, 0]$. Let E be the linear space of (equivalence classes of) strongly measurable functions φ from $(-\infty, 0]$ to X such that φ is continuous on $[-r, 0]$ and $p\varphi$ is integrable on $(-\infty, -r)$. Then E becomes a Banach space under the norm

$$\|\varphi\|_E = \max\{\sup_{-r\leq\xi\leq 0} p(\xi)\|\varphi(\xi)\|, \int_{-\infty}^{-r} p(\xi)\|\varphi(\xi)\|d\xi\}.$$

This space fails to satisfy (E.1)(c). However if $\|\varphi_n - \varphi\|_E \to 0$, then there exists a subsequence (φ_{n_k}) such that

$$\left\|\varphi_{n_k}(\theta) - \varphi(\theta)\right\| \to 0, \quad \text{a.e.} \quad \theta \in (-\infty, -r].$$

An inspection of the method of proof in [20] and [15] shows that this still yields the translation property of $U(t, s)$. Thus Theorem 2.2 holds for this initial history space as well .

2.4 Characterizing the closure of $D(A(t))$

In this section we characterize the closure of $D(A(t))$ for the operator $A(t)$ defined in (2.3). Let us assume the following conditions :

(B.1′) $(B(t))_{t\geq 0}$ is a family of $\alpha-$accretive operators.

(B.2′) $F(t, .)$, $t \geq 0$ is a Lipschitz continuous function on $\hat{E}(t)$ with Lipschitz constant M.

(B.3′) For $x \in \hat{X}(t)$, $\psi \in \hat{E}(t)$, and $\lambda > 0$ with $\lambda\omega < 1$, and $\omega = \max\{0, M + \alpha\}$,

(i) if $\varphi^{\psi}_{x,\lambda} \in E$ is the solution to $\begin{cases} \varphi - \lambda\varphi' = \psi \\ \varphi(0) = x \end{cases}$, then $\varphi^{\psi}_{x,\lambda} \in \hat{E}(t)$, and, moreover,

(ii) $\quad \left[\psi(0) + \lambda F(t, \varphi^{\psi}_{x,\lambda})\right] \in (I + \lambda B(t))D(B(t) \cap \hat{X}(t)).$

Lemma 2.12. *If (B.1′)-(B.3′) are satisfied then for* $\lambda > 0$ *such that* $\lambda\omega < 1$

$$cl(D(A(t))) \subseteq \hat{E}(t) \subseteq R(I + \lambda A(t)).$$

Proof. Let $\varphi \in \hat{E}(t)$. Consider $T : \hat{X}(t) \to \hat{X}(t)$ defined by

$$T(x) = J^{B(t)}_{\lambda}(\varphi(0) + \lambda F(t, \varphi_x)).$$

Then T is a contraction, and therefore has a unique fixed point x_0. Now following the argument in the proof of Proposition 2.3 one can see that $\varphi^{\varphi}_{x_0,\lambda} \in D(A(t))$ and $(I + \lambda A(t))\varphi^{\varphi}_{x_0,\lambda} = \varphi$.

Proposition 2.13. *Under the assumptions of Lemma 2.12*

$$cl(D(A(t))) = \left\{\varphi \in \hat{E}(t) |\ \ \varphi(0) \in cl(D(B(t)))\right\}.$$

Proof. Let $\varphi \in \hat{E}(t)$ such that $\varphi(0) \in cl(D(B(t)))$. Choose $a_n \in D(B(t))$ such that $\|\varphi(0) - a_n\| \to 0$ as $n \to \infty$. Let $\lambda_n \downarrow 0$, then by Lemma 2.12 for n large enough we may define $\varphi_n = J_{\lambda_n}^{A(t)}\varphi$. Using Lemma 2.5, and (E.2) we have

$$\|\varphi_n - \varphi\| \leq \left\|J_{\lambda_n}^{A_0}(\varphi - \varphi(0)) - (\varphi - \varphi(0))\right\| + \left\|(J_{\lambda_n}^{A(t)}\varphi)(0) - \varphi(0)\right\|. \tag{2.33}$$

Since $\varphi - \varphi(0) \in E_0$, the first term in (2.33) tends to 0 as $n \to \infty$.

$$\begin{aligned}\left\|(J_{\lambda_n}^{A(t)}\varphi)(0) - \varphi(0)\right\| = \left\|J_{\lambda_n}^{B(t)}\big(\lambda_n F(t,\varphi_n) + \varphi(0)\big) - \varphi(0)\right\| &\leq \\ \left\|J_{\lambda_n}^{B(t)}\big(\lambda_n F(t,\varphi_n) + \varphi(0)\big) - (\lambda_n F(t,\varphi_n) + \varphi(0))\right\| + \lambda_n\|F(t,\varphi_n)\| &\leq \\ \frac{2-\lambda_n\alpha}{1-\lambda_n\alpha}\|\lambda_n F(t,\varphi_n) + \varphi(0) - a_m\| + \frac{\lambda_n}{1-\lambda_n\alpha}\|b_m\| + \lambda_n\|F(t,\varphi_n)\| &\end{aligned}$$

for all $m \in \mathbb{N}$ and $b_m \in B(t)a_m$. Here we have used Lemma 1.4. Using the Lipschitz property of F in the last inequality we have

$$\begin{aligned}\left\|J_{\lambda_n}^{A(t)}\varphi(0) - \varphi(0)\right\| &\leq \frac{2-\lambda_n\alpha}{1-\lambda_n\alpha}\|\varphi(0) - a_m\| + \frac{\lambda_n}{1-\lambda_n\alpha}\|b_m\| \\ &+ \frac{3-2\lambda_n\alpha}{1-\lambda_n\alpha}\left(\lambda_n M\|\varphi_n - \varphi\| + \lambda_n\|F(t,\varphi)\|\right).\end{aligned} \tag{2.34}$$

Now combining (2.33) and (2.34), we have

$$\limsup_{n\to\infty}\|\varphi_n - \varphi\| \leq 2\|\varphi(0) - a_m\| \quad \text{for all} \quad m \in \mathbb{N},$$

and therefore $\varphi_n \to \varphi$. Since $\varphi_n \in D(A(t))$, we conclude that $\varphi \in clD(A(t))$.

Corollary 2.14. *If in Theorem 2.2, (B.2′) and (B.3′) are also satisfied, then for all* $\varphi \in \hat{E}(s)$ *such that* $\varphi(0) \in cl(D(B(s)))$ *the conclusions of Theorem 2.2 hold.*

For some special choices of $\hat{E}(t)$, $cl(D(A(t)))$ can also be determined under the conditions of Theorem 2.2.

Remark 2.15. Let $\hat{E}(t) = \hat{E}_0(t) = \{\varphi \in E \,|\, \varphi(0) \in cl(\hat{X}(t) \cap D(B(t)))\}$, or $\hat{E}(t) = \{\varphi \in E \,|\, \varphi(0) \in \hat{X}(t) \cap cl(D(B(t)))\}$, then under the assumptions of Theorem 2.2, $cl(D(A(t))) = \hat{E}_0(t)$. (The proof follows from the same argument as in Proposition 3.17 of Chapter 3.)

Remark 2.16. Following [60, Remark 2.8], we now note some relationships between the families $\hat{E}(t)$ and $\hat{X}(t)$.
(i) If (B.3$'$)(i) is satisfied for $\hat{X}(t) \subset X$ and $\hat{E}(t) \subset E$, then

$$\hat{X}(t) \subset \hat{E}(t)(0) = \{\varphi(0) \mid \varphi \in \hat{E}(t)\}.$$

(ii) If both conditions (B.3$'$)(i) and (B.3$'$)(ii) are satisfied for $\hat{X}(t) \subset X$ and $\hat{E}(t) \subset E$, then $cl(D(A(t)))(0) \subset \hat{X}(t)$.
Part (i) of the above remark is obvious. For (ii), let $x \in \hat{X}(t)$, and choose $\lambda_n \downarrow 0$ such that $\lambda_n\omega < 1$ for all $n \in \mathbb{N}$. Take $\psi \in cl(D(A(t)))$, then (B.3$'$)(i) yields that $\varphi^{\psi}_{x,\lambda_n}$ belongs to $\hat{E}(t)$ and $\left\|\varphi^{\psi}_{x,\lambda_n}\right\| \leq \max\{\|x\|, \|\psi\|\}$, $n \in \mathbb{N}$. Using (B.2$'$)(ii), there exists a sequence $(x_n) \in \hat{X}(t) \cap D(B(t))$ such that $\psi(0) + \lambda_n F(t, \varphi^{\psi}_{x,\lambda_n}) \in (I + \lambda_n B(t))x_n$. Since $\psi(0) \in cl(D(B(t)))$, we can choose $[a_m, b_m] \in B(t)$ such that $a_m \to \psi(0)$, as $m \to \infty$. Now using Lemma 1.4 we have

$$\begin{aligned} \|x_n - \psi(0)\| = \left\|J^{B(t)}_{\lambda_n}\big(\psi(0) + \lambda_n F(t, \varphi^{\psi}_{x,\lambda_n})\big) - \psi(0)\right\| \ \leq \\ \frac{2 - \lambda_n\omega}{1 - \lambda_n\omega}\left\|\psi(0) + \lambda_n F(t, \varphi^{\psi}_{x,\lambda_n}) - a_m\right\| + \frac{\lambda_n}{1 - \lambda_n\omega}\|b_m\| \ + \ \lambda_n\left\|F(t, \varphi^{\psi}_{x,\lambda_n})\right\| \end{aligned} \tag{2.35}$$

for all $m \in \mathbb{N}$. Since the sequence $(\varphi^{\psi}_{x,\lambda_n})$ is bounded we have

$$\limsup_{n\to\infty} \|x_n - \psi(0)\| \leq 2\|\psi(0) - a_m\| \quad \text{for all} \quad m \in \mathbb{N},$$

which implies $\|x_n - \psi(0)\| \to 0$, and, therefore $\psi(0) \in \hat{X}(t)$.

Corollary 2.17. *If in addition to (B.1)-(B.3), also (B.3$'$) is fulfilled, in Theorem 2.2 we also have invariance of $\hat{X}(t)$; $U(t,s)\varphi(0) = u_{\varphi}(t) \in \hat{X}(t)$ for all $t \geq s$.*

Remarks 2.18. 1. In the case that the family $B(t)$ is m- accretive, and the operators $F(t,\cdot)$ are globally defined, with the choice of $\hat{X}(t) = X$, and $\hat{E}(t) = E$ assumptions (B.3) and (B.3$'$) are automatically fulfilled. If $F(t,.)$ is globally Lipschitz and $\|F(t,\varphi) - F(\tau,\varphi)\| \leq \|k(t) - k(\tau)\|L(\|\varphi\|)$ for a continuous function k and a bounded function L, then (B.2) and (B.2$'$) are also satisfied, with $\hat{E}(t) = E$. Thus Theorem 2.2 extends the previous related works on (FDE) in [6, 7, 12, 13, 14, 15, 16, 29, 30, 31, 32, 33, 34, 35, 36, 45, 53, 57, 58, 69, 70, 71, 72, 73, 74].

2. The result includes an assertion on flow invariance: by (2.19), and the fact that for $\varphi \in cl(D(A(s)))$, $U(t,s)\varphi$ remains in $cl(D(A(t)))$, automatically $(u_{\varphi})_t \in \hat{E}(t)$, for all $t \geq s$.

3. If we choose $\hat{E}(t) = \{\varphi \in E \,|\, \varphi(0) \in \hat{X}(t)\}$, then (B.3)($i$) is satisfied. However as we will see, in the examples condition (B.2) can not be easily verified.
In the concrete examples for $\hat{E}(t) = \{\varphi \in E \,|\, \varphi(s) \in \hat{X}(t), s \in I\}$, condition (B.2) is fulfilled but to show (B.3)(i), we need the family $\hat{X}(t)$ to be nondecreasing and convex.

2.5 Diffusive population models with delay in the birth process

We look at the time dependent population equation with delay

$$\begin{cases} \dot{u}(x,t) - d(t)\Delta u(x,t) \ni a(t)u(x,t)\big[1 - b(t)u(x,t) - \\ \qquad\qquad\qquad\qquad \int_{-1}^{0} u(x,t+r(s))d\eta_t(s)\big], \quad t \geq 0 \\ u_{|[-R,0]} = \varphi \\ +\text{boundary conditions}, \end{cases} \tag{2.36}$$

where $a, b, d : \mathbb{R}^+ \to \mathbb{R}^+$ are bounded continuous functions, such that $0 < d_0 \leq d(t)$, $0 < a_0 \leq a(t) \leq a_1$ and $0 < b_0 \leq b(t) \leq b_1$. The map $t \mapsto \eta_t$ from $\mathbb{R}^+$ to $M^+([-1,0])$ is continuous, with $b(t) + \|\eta_t\| = 1$, and $r : [-1,0] \to [-R,0]$ is a continuous delay function, and X is an appropriate state space of real valued functions on $\Omega \subset \mathbb{R}^N$ open. In particular, this equation serves as a model for the density of red blood cells in an animal. This and related concrete population models (see below) have been considered by various authors under both Dirichlet and (linear) Neumann boundary conditions. The state spaces considered in these works have been restricted to function spaces on Ω that are invariant under products, such as, for $\Omega \subset \mathbb{R}^N$ bounded, either $C(\bar{\Omega})$ or $W^{2,p}(\Omega)$, with $N < p < \infty$; for a partial list of references, compare [26, 43, 46, 67, 80]. The reason for this restriction are the quadratic terms in the history responsive operator

$$F(t,\varphi) = a(t)\varphi(0)\left[1 - b(t)\varphi(0) - \int_{-1}^{0} \varphi(r(s))d\eta_t(s)\right].$$

However, the natural state space for population models obviously is $L^1(\Omega)$. For further references and more details in this direction, see Ruess [58].
The results of this chapter now make it possible to place this model in the context of the natural state space $L^1(\Omega)$. Moreover, the Laplacian can be replaced by more general, possibly nonlinear diffusion operators in divergence form.

Definition 2.19. 1. An operator A in $L^1(\Omega)$ is called completely accretive if for all $\lambda > 0$, and $[u,v], [\tilde{u},\tilde{v}] \in A$

$$\int_\Omega j(u-\tilde{u}) \leq \int_\Omega j(u-\tilde{u}+\lambda(v-\tilde{v})), \qquad j \in \mathbf{J}_0, \tag{2.37}$$

where $\mathbf{J}_0 = \{j : \mathbb{R} \to [0,\infty) : j$ is convex lower-semicontinuous, and $j(0) = 0\}$.
2. A is called m-completely accretive in $L^1(\Omega)$ if A is completely accretive and $R(I+\lambda A) = L^1(\Omega)$, for $\lambda > 0$.

Remark 2.20. If A is a completely accretive operator in $L^1(\Omega)$, then by choosing $j(r)$ to be r^+, and $|r|$ in (2.37), we see that J_λ^A are order preserving and contraction in $\|.\|_1$. Moreover if $u, \tilde{u} \in R(I+\lambda A) \cap L^\infty(\Omega)$, then $\left\|J_\lambda^A u - J_\lambda^A \tilde{u}\right\|_\infty \leq \|u-\tilde{u}\|_\infty$; here one can take $j(r) = (|r| - \|u-\tilde{u}\|_\infty)^+$. For more details about completely accretive operators we refer to [2, 75].

Prominent examples of m-completely accretive operators are diffusion operators of the form

$$-div\; a(.,\; grad\; u) + \tilde{\beta}(u) + \quad \text{Dirichlet boundary conditions,}$$

with $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following conditions:

(H1) $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Caratheodory function, i.e., measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^n$, and continuous in $\xi \in \mathbb{R}^n$ for a.e. $x \in \Omega$; $a(\cdot,0) = 0$;

(H2) "monotonicity condition": $(a(x,\xi) - a(x,\hat{\xi})) \cdot (\xi - \hat{\xi}) \geq 0$ for all $\xi, \hat{\xi} \in \mathbb{R}^n$, a.e. $x \in \Omega$;

(H3) "coerciveness condition": $a(x,\xi)\cdot\xi \geq \lambda_0|\xi|^p - a_0(x)$ for all $\xi \in \mathbb{R}^n, |\xi| \geq R_0$, a.e. $x \in \Omega$, where $1 < p < \infty$, $a_0 \in L^1(\Omega)$, $\lambda_0 > 0$, $R_0 \geq 0$;

(H4) $|a(x,\xi)| \leq b_0(x) + C_0|\xi|^{p-1}$ for all $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$, where $b_0 \in L^{p'}(\Omega) + L^\infty(\Omega)$, $1/p + 1/p' = 1$, $C_0 \geq 0$,

or more generally

$$-div\; a(.,grad\; u); \qquad -a(.,grad\; u).n \in \beta(u) \quad \text{on} \quad \partial\Omega,$$

with a as above and $\beta \subset \mathbb{R}^2$ a maximal monotone graph such that $0 \in \beta(0)$ (cf.[76, 77, 78, 79]).
We now consider the model (2.36) in $L^1(\Omega)$ and with $-\Delta$ replaced by any such more general diffusion operator, or by just any family $B(t)$ of m-completely accretive operators.

Proposition 2.21. *Let Ω be an open subset of $\mathbb{R}^N$, $X = L^1(\Omega)$, and let $B(t) \subset X \times X$ be a family of m-completely accretive operators with $0 \in B(t)0$, for all $t \geq 0$. If in addition the family $B(t)$ satisfies an inequality of the form (2.1) and $a(t)$ is non increasing, then for all $\alpha \geq \max\{\frac{a(0)}{b_0}, a_0\}$, the family $\hat{X}(t) \subset X$, $\hat{X}(t) = \{x \in X \mid 0 \leq x(\omega) \leq \frac{\alpha + a(t)}{a(t)} \quad a.e. \quad \omega \in \Omega\}$, is invariant for the delay equation*

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni a(t)u(t)\left[1 - b(t)u(t) - \int_{-1}^{0} u(t + r(s))d\eta_t(s)\right] & t \geq 0 \\ u_{|[-R,0]} = \varphi \end{cases} \tag{2.38}$$

in the following sense; for any $\varphi \in C([-R, 0]; X)$ with $\varphi(s) \in \hat{X}(0), s \in [-R, 0]$, and $\varphi(0) \in cl(D(B(0)))$, if $u \in C(\mathbb{R}^+; X)$ is the unique global (mild) solution of (2.38) as in Theorem 2.2, then $u(t) \in \hat{X}(t)$ for all $t \geq 0$.

Proof. Set $\beta(t) = \frac{\alpha + a(t)}{a(t)}$, for $\alpha > 0$ to be specified later. Let $\hat{X}(t) = \{x \in X \mid 0 \leq x(\omega) \leq \beta(t) \quad \text{a.e.} \quad \omega \in \Omega\}$, and

$$\hat{E}(t) = \{\varphi \in C([-R, 0]; X) \,|\, \varphi(s) \in \hat{X}(t),\, s \in [-R, 0]\}.$$

We now consider the equation (2.38) in the form of (FDE) with $F(t, .) : \hat{E}(t) \to X$, defined by $F(t, \varphi) = a(t)\varphi(0)\left[1 - b(t)\varphi(0) - \int_{-1}^{0} \varphi(r(s))d\eta_t(s)\right]$.
As in [62, Section 4], we replace the operator $B(t)$ in (2.38) by $B_\alpha(t) := \alpha I + B(t)$, and accordingly, change the history-responsive operator to $F_\alpha(t, \varphi) := F(t, \varphi) + \alpha\varphi(0)$. Let $x \in \hat{X}(t + \lambda)$, $\psi \in \hat{E}(t)$, and $\lambda > 0$. Since $a(t)$ are non increasing, $\int_s^0 e^{-\xi/\lambda}\psi d\xi \in \lambda(e^{-s/\lambda} - 1)\hat{X}(t + \lambda)$ for each $s \in I$, [52, Theorem 3.27]. Hence

$$\varphi_x(s) = e^{s/\lambda}\big(x + \frac{1}{\lambda}\int_s^0 e^{-\xi/\lambda}\psi(\xi)d\xi\big),$$

belongs to $\hat{X}(t + \lambda)$ for $s \in [-R, 0]$.
Let $\varphi \in \hat{E}(t)$ and $\psi \in \hat{E}(\tau), 0 \leq \tau \leq t$. Some elementary computations show that

$$\begin{aligned} \|F_\alpha(t, \varphi) - F_\alpha(\tau, \psi)\| \leq M_\alpha\|\varphi - \psi\|_\infty + \\ L(\|\psi\|_\infty)\big(|a(t) - a(\tau)| + \|\eta_t - \eta_\tau\| + |a(t)b(t) - a(s)b(s)|\,\big) \end{aligned} \tag{2.39}$$

where $L : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $L(x) = x + \beta(2 + a(0))x$ with $\beta = \sup\{\beta(\tau) \,|\, \tau \geq 0\}$, and $M_\alpha = \alpha + a(0)(1 + 2\beta)$. If we choose

$$\alpha \geq \max\{\frac{a(0)}{b_0}, a_0\},$$

then

$$0 \leq \frac{1}{1+\lambda\alpha}\left[\psi(0) + \lambda F_\alpha(t+\lambda, \varphi_x)\right] \leq \beta(t+\lambda) \quad \text{a.e. on} \quad \Omega. \tag{2.40}$$

Using the fact that $B(t)$'s have order-preserving resolvents (defined on all of X) that also contract in the $L^\infty-$norm (together with $0 \in B(t)0$), we conclude from (2.40) that

$$0 \leq (I + \frac{1}{1+\lambda\alpha} B(t+\lambda))^{-1}\{\frac{1}{1+\lambda\alpha}\left[\psi(0) + \lambda F_\alpha(t+\lambda, \varphi_x)\right]\} \leq \beta(t+\lambda) \quad \text{a.e. on} \quad \Omega,$$

and this shows that

$$0 \leq (I + \lambda B_\alpha(t+\lambda))^{-1}\{\psi(0) + \lambda F_\alpha(t+\lambda, \varphi_x)\} \leq \beta(t+\lambda) \quad \text{a.e. on} \quad \Omega. \tag{2.41}$$

Therefore $(B.1)$-$(B.3)$ are satisfied in our setting. The assertions of Proposition (2.21) can now be read from Theorem (2.2).

Remark 2.22. The condition on $a(t)$ to be non-increasing is needed to get the optimal estimate for $\beta(t)$. However, under the condition of Proposition 2.21, if $a(.)$ is a bounded continuous function on $\mathbb{R}^+$ such that $0 < a_0 \leq a(t) \leq a_1$, $t \geq 0$, then we may choose;

$$\hat{X}(t) = \{x \in X \,|\, 0 \leq x(\omega) \leq \frac{\alpha}{2a_1} \quad \text{a.e.} \quad \omega \in \Omega\},$$

and
$$\hat{E}(t) = \{\varphi \in C([-R, 0]; X) \,|\, \varphi(s) \in \hat{X}(t),\ -R \leq s \leq 0\},$$
with $\alpha \geq \frac{2a_1^2}{a_0 b_0}$.

Existence and flow-invariance results corresponding to those of Proposition 2.21 for the model (2.38) can also be achieved for the related model

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni u(t)\left[1 + a(t)u(t) - b(t)(u(t))^2 \right. \\ \qquad\qquad\qquad \left. -(1 + a(t) - b(t)) \int_{-r}^{0} f(s)u(t+s)ds\right], \quad t \geq 0 \\ u_{|[-r,0]} = \varphi \end{cases} \tag{2.42}$$

for Ω open in $\mathbb{R}^N$, $B(t) \subset L^1(\Omega) \times L^1(\Omega)$ m–completely accretive (such as (2.36) above), $a, b : \mathbb{R}^+ \to \mathbb{R}^+$, are bounded continuous functions such that $b(t) < 1 + a(t), t \in \mathbb{R}^+$, and $f \in L^1((-R, 0))$ nonnegative with $\|f\|_1 = 1$. For Ω bounded, a, b constant, and $B(t) = -\Delta$ with $0-$Neumann boundary conditions, and state space $C(\bar{\Omega})$, see [23, 51].

Also, both models (2.38) and (2.42) can be extended to infinite delays and, more importantly, to temporal averages being replaced by spatio-temporal averages over the past history, such as

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni a(t)u(t)\left[1 - b(t)u(t) \right. \\ \qquad\qquad\qquad\qquad \left. - \int_{-\infty}^{t} \int_{\Omega} g(\cdot - y, t - s)u(s)(y)dyds\right], \quad t \geq 0 \\ u_{|[-\infty,0]} = \varphi \end{cases} \tag{2.43}$$

and

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni u(t)\left[1 + a(t)u(t) - b(t)(u(t))^2 \right. \\ \qquad\qquad \left. -(1 + a(t) - b(t)) \int_{-\infty}^{t} \int_{\Omega} g(\cdot - y, t - s)u(s)(y)dyds\right], \quad t \geq 0 \\ u_{|[-\infty,0]} = \varphi \end{cases} \tag{2.44}$$

with $g \in L^1(\mathbb{R}^N \times (0, \infty))$ nonnegative suitably chosen, and Ω and $B(t)$ as in (2.38) and (2.42). For the discussion of model (2.44) for Ω bounded, a, b constant, and $B(t) = -\Delta$, together with further relevant references, the reader is referred to [8].

2.6 Initial history spaces of L^1 type

In the context of (FDE) with initial history space $E = L^1(I; X)$, it is necessary to take the initial value $u(s) = x \in X$ as an additional datum. Thus , instead of working in the context of $E = L^1(I; X)$, one needs to work with the product space $E_X = L^1(I; X) \times X$, $\|[\varphi, h]\| = \max\{\|\varphi\|_1, \|h\|\}$.

Following [58, Section 4], throughout this section, we shall take the initial history space E, $E = L^1(\mathbb{R}^-, \nu, X)$ with $\nu = p(.)d\lambda$, where the Lebesgue measurable function $p : \mathbb{R}^- \to (0, 1]$ is chosen such that, for some $\mu \geq 0$,

$$p(s)e^{\mu s} \quad \text{is nondecreasing on} \quad \mathbb{R}^-, \quad \text{and} \quad p(0) = 1. \tag{2.45}$$

We then start from the space $E_X = L^1(\mathbb{R}^-, \nu, X) \times X$, with norm $\|[\varphi, h]\| = \max\{\|\varphi\|_1, \|h\|\}$, and denote by π_1 and π_2 the projections of E_X onto its first and second component, respectively.

(FDE) will be studied in the following form.

$$(\text{FDE})_1 \quad \begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t, u(t)), \quad 0 \leq s \leq t \\ u_s = \varphi, \quad u(s) = h \in X. \end{cases}$$

For a fixed $T > 0$, the assumptions (B.1)–(B.3) of Section 2 are modified as follows:

$(B1)_1$ $(B(t))_{0\le t\le T}$ is a family of operators $B(t)\subset X\times X$ such that there exist $\alpha\in\mathbb{R}$, a continuous function $f:[0,T]\to X$, and a nondecreasing bounded function (on bounded sets) $L_1:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $[x_i,y_i]\in B(t_i), i\in\{1,2\}$, and $0\le t_2\le t_1\le T$,

$$(1-\lambda\alpha)\|x_1-x_2\|\le\|x_1-x_2+\lambda(y_1-y_2)\|+\lambda\|f(t_1)-f(t_2)\|L_1(\|x_2\|), \tag{2.46}$$

for all $\lambda>0$ with $\lambda\alpha<1$.

$(B2)_1$ $\hat{E}(t)$, and $\hat{X}(t)$, $t\in[0,T]$, are closed subsets of E, and X such that
(i) $F:\bigcup_{t\in[0,T]}\{t\}\times\hat{E}(t)\times\hat{X}(t)\to X$, is continuous and bounded on bounded sets.
(ii) There exists $M>0$ such that for $\varphi\in\hat{E}(t_1)$ and $\psi\in\hat{E}(t_2)$ continuous, $0\le t_2\le t_1\le T$, with $\varphi(0)\in\hat{X}(t_1)$, $\psi(0)\in\hat{X}(t_2)$, and $\|[\varphi,\varphi(0)]-[\psi,\psi(0)]\|=\|\varphi(0)-\psi(0)\|$:

$$\begin{aligned}\|F(t_1,\varphi,\varphi(0))-F(t_2,\psi,\psi(0))\| &\le M\|\varphi(0)-\psi(0)\|\\ &+\ \|g(t_1)-g(t_2)\|L_2(\|[\psi,\psi(0)]\|)\end{aligned}$$

where $g:[0,T]\to X$ is a continuous function and $L_2:\mathbb{R}^+\to\mathbb{R}^+$ is a bounded (on bounded sets) function.

$(B3)_1$ For $x\in\hat{X}(t+\lambda)$, $\psi\in\hat{E}(t)$, $\lambda>0$ with $\lambda\omega<1$, and $\omega:=\max\{1-\mu,M+\alpha\}$,
(i) if $\varphi^{\psi}_{x,\lambda}\in E$ is the solution to

$$\begin{cases}\varphi-\lambda\varphi'=\psi,\\ \varphi(0)=x,\end{cases}$$

then $\varphi^{\psi}_{x,\lambda}\in\hat{E}(t+\lambda)$, and, moreover for all $k\in\hat{X}(t)$

(ii) $$\big(k+\lambda F(t+\lambda,\varphi^{\psi}_{x,\lambda})\big)\in(I+\lambda B(t+\lambda))(D(B(t+\lambda))\cap\hat{X}(t+\lambda)).$$

Accordingly, associated with $(FDE)_1$, we define the family of operators in E_X by

$$\begin{cases}D(A(t))=\big\{[\varphi,h]\in\hat{E}(t)\times\hat{X}(t)\ :\varphi\quad\text{locally absolutely continuous on}\quad\mathbb{R}^-,\\ \quad\text{differentiable a.e.}\quad s\in(-\infty,0],\quad\varphi'\in E,\quad\varphi(0)=h\in D(B(t))\big\}\\ A(t)[\varphi,h]:=[-\varphi',-F(t,\varphi,h)+B(t)h],\quad[\varphi,h]\in D(A(t)),\end{cases} \tag{2.47}$$

and consider the following statements:
$(S1)_1$ $A(t)$ generates an evolution operator $U(t,s)$, $U(t,s):cl(D(A(s)))\to cl(D(A(t)))$

of type γ, with $\gamma = \max\{1-\mu, \alpha + M\}$:

$$\|U(t,s)[\varphi_1, h_1] - U(t,s)[\varphi_2, h_2]\| \leq e^{\gamma(t-s)}\|[\varphi_1, h_1] - [\varphi_2, h_2]\|$$

for all $0 \leq s \leq t \leq T$, $[\varphi_i, h_i] \in cl(D(A(s)))$, $i \in \{1,2\}$.

(S2)$_1$ If $[\varphi, h] \in cl(D(A(s)))$, and $u(\varphi, h) : (I+s) \cup [s,T] \to X$ is defined by

$$u(\varphi,h)(t) = \begin{cases} \varphi(t-s), & I \ni t-s \leq 0 \\ \pi_2 U(t,s)[\varphi,h], & t \geq s \end{cases} \tag{2.48}$$

where $U(t,s)$ is the evolution operator generated in (S1)$_1$, then

$$\pi_1 U(t,s)[\varphi,h] = (u(\varphi,h))_t, \qquad s \leq t \leq T.$$

(S3)$_1$ For $[\varphi, h] \in cl(D(A(s)))$, the function $u(\varphi, h)$ defined in (S2)$_1$ is a mild solution to (FDE)$_1$ with $(u(\varphi,h))_t \in \hat{E}(t)$ and $u(\varphi,h)(t) \in \hat{X}(t)$, $t \geq s$.

In the present context, the following theorem is the analogue of Theorem 2.2.

Theorem 2.23. *Let (B1)$_1$–(B3)$_1$ are satisfied. Then given $s \in [0,T)$ and $[\varphi, h] \in cl(D(A(s)))$ statements (S1)$_1$, (S2)$_1$ and (S3)$_1$hold.*

Proof. According to Theorem 1.7, to show that $A(t)$ generates an evolution operator $U(t,s)$, $U(t,s) : cl(D(A(s))) \to cl(D(A(t)))$, it is enough to prove that (A.1), (A.2), and (A.3) are satisfied in our setting.

(A1)$_1$ Let $\lambda > 0$, and $\lambda\gamma < 1$ with $\gamma = \max\{1-\mu, \alpha + M\}$.
Take $[-\varphi_i', k_i] \in A(t_i)[\varphi_i, h_i]$, $i \in \{1,2\}$, and $s \leq t_2 \leq t_1 \leq T$. We consider two cases:
1. If $\|h_1 - h_2\| \leq \|\varphi_1 - \varphi_2\|_1$, then

$$\|[\varphi_1, h_1] - [\varphi_2, h_2]\| = \|\varphi_1 - \varphi_2\|_1. \tag{2.49}$$

Set $\varphi_i - \lambda\varphi_i' = \psi_i$, $i \in \{1,2\}$. Then

$$(\varphi_1 - \varphi_2) + \lambda(-\varphi_1' + \varphi_2') = (\psi_1 - \psi_2). \tag{2.50}$$

Moreover,

$$\|\varphi_1 - \varphi_2\|_1 = \int_{-\infty}^{0} \left\| e^{s/\lambda}(h_1 - h_2) + \frac{e^{s/\lambda}}{\lambda}\int_s^0 e^{-\xi/\lambda}(\psi_1(\xi) - \psi_2(\xi))d\xi \right\| p(s)ds \quad \leq$$

$$\|h_1 - h_2\| \int_{-\infty}^{0} e^{(s/\lambda + \mu s)}e^{-\mu s}p(s)ds + \frac{1}{\lambda}\int_{-\infty}^{0} e^{s/\lambda}\int_s^0 e^{-\xi/\lambda}\|\psi_1(\xi) - \psi_2(\xi)\|d\xi p(s)ds \quad \leq$$

$$\|h_1 - h_2\| \int_{-\infty}^{0} e^{(s/\lambda + \mu s)}ds + \frac{1}{\lambda}\int_{-\infty}^{0} e^{-\xi/\lambda}\|\psi_1(\xi) - \psi_2(\xi)\| \int_{-\infty}^{\xi} e^{s/\lambda}e^{\mu s}e^{-\mu s}p(s)dsd\xi \quad \leq$$

$$\frac{\lambda}{1+\lambda\mu}\|h_1-h_2\| + \frac{1}{1+\lambda\mu}\int_{-\infty}^{0} e^{-\xi/\lambda}\|\psi_1(\xi)-\psi_2(\xi)\|e^{(\xi/\lambda+\mu\xi)}e^{-\mu\xi}p(\xi)d\xi \quad \leq \quad \frac{\lambda}{1+\lambda\mu}\|\varphi_1-\varphi_2\|_1 + \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_1.$$

Here we have used (2.45). Then we conclude from (2.49), and (2.50) that

$$(1+\lambda\mu-\lambda)\|[\varphi_1,h_1]-[\varphi_2,h_2]\| \quad \leq \quad \|[\varphi_1,h_1]-[\varphi_2,h_2]+\lambda([-\varphi_1',k_1]-[-\varphi_2',k_2])\|.$$

Since $\gamma \geq (1-\mu)$, the above inequality clearly implies inequality (2.52).
2. $\|\varphi_1-\varphi_2\|_1 \leq \|h_1-h_2\|$. In this case

$$\|[\varphi_1,h_1]-[\varphi_2,h_2]\| = \|h_1-h_2\|. \tag{2.51}$$

We also note that $k_i \in -F(t_i,\varphi_i,h_i)+B(t_i)h_i$, and therefore $[h_i, k_i+F(t_i,\varphi_i,h_i)] \in B(t_i)$, $i \in \{1,2\}$. Therefore, (2.46) implies that

$$\begin{aligned}(1-\lambda\alpha)\|h_1-h_2\| \quad &\leq \quad \big\|h_1-h_2+\lambda\big(k_1+F(t_1,\varphi_1,h_1)-k_2-F(t_2,\varphi_2,h_2)\big)\big\| \\ &+ \quad \lambda\|f(t_1)-f(t_2)\|L_1(\|h_2\|).\end{aligned}$$

Now (2.51), allows us to apply $(\mathrm{B2})_1(ii)$ to the last inequality. Hence

$$\begin{aligned}(1-\lambda\alpha)\|h_1-h_2\| \quad &\leq \quad \|h_1-h_2+\lambda(k_1-k_2)\|+\lambda M\|h_1-h_2\| \\ &+ \quad \lambda\|g(t_1)-g(t_2)\|L_2\big(\|[\varphi_2,h_2]\|\big) \\ &+ \quad \lambda\|f(t_1)-f(t_2)\|L_1(\|[\varphi_2,h_2]\|).\end{aligned}$$

But $\|h_1-h_2+\lambda(k_1-k_2)\| \leq \|[\varphi_1,h_1]-[\varphi_2,h_2]+\lambda([-\varphi_1',k_1]-[-\varphi_2',k_2])\|$, thus (2.51) and the choice for γ imply that

$$\begin{aligned}(1-\lambda\gamma)\|[\varphi_1,h_1]-[\varphi_2,h_2]\| \quad &\leq \\ \|[\varphi_1,h_1]-[\varphi_2,h_2]+\lambda([-\varphi_1',k_1]-[-\varphi_2',k_2])\| \quad &+ \\ \lambda\big(\|g(t_1)-g(t_2)\|+\|f(t_1)-f(t_2)\|\big)L(\|[\varphi_2,h_2]\|)&\end{aligned} \tag{2.52}$$

which is the desired inequality. As in Section 2.3 we define

$$H(t_1,t_2) = \|g(t_1)-g(t_2)\|+\|f(t_1)-f(t_2)\|. \tag{2.53}$$

$(\mathrm{A2})_1$ Let $t \geq s$, and $\lambda > 0$ with $\lambda\gamma < 1$. We shall prove that

$$cl(D(A(t))) \subseteq \hat{E}(t)\times\hat{X}(t) \subseteq R(I+\lambda A(t+\lambda)). \tag{2.54}$$

Let $[\psi, k] \in \hat{E}(t) \times \hat{X}(t)$. By (B3)$_1(ii)$ we may define;

$$T : \hat{X}(t+\lambda) \to \hat{X}(t+\lambda) \quad \text{by} \quad T(x) = J_\lambda^{B(t+\lambda)}\big(k + \lambda F(t+\lambda, \varphi^\psi_{x,\lambda}, x)\big).$$

Then

$$\|Tx - Ty\| \leq \frac{\lambda}{1-\lambda\alpha}\Big\|F(t+\lambda, \varphi^\psi_{x,\lambda}, x) - F(t+\lambda, \varphi^\psi_{y,\lambda}, y)\Big\| \leq \frac{\lambda M}{1-\lambda\alpha}\|x-y\|.$$

Here we have used that $\Big\|\varphi^\psi_{x,\lambda} - \varphi^\psi_{y,\lambda}\Big\|_1 \leq \|x-y\|$. Indeed,

$$\begin{aligned} \Big\|\varphi^\psi_{x,\lambda} - \varphi^\psi_{y,\lambda}\Big\|_1 &= \int_{-\infty}^0 \big\|e^{s/\lambda}(x-y)\big\|p(s)ds \\ &= \|x-y\| \int_{-\infty}^0 e^{(s/\lambda+\mu s)}e^{-\mu s}p(s)ds \\ &= \frac{\lambda}{1+\lambda\mu}\|x-y\| \leq \|x-y\|, \end{aligned}$$

where the last inequality follows from our choice for γ. Since $\lambda(M+\alpha) < 1$, we conclude that T is a strict contraction, and therefore there exists a unique point $z \in \hat{X}(t+\lambda)$, such that $z = J_\lambda^{B(t+\lambda)}\big(k + \lambda F(t+\lambda, \varphi^\psi_{z,\lambda}, z)\big)$. Thus $z = \varphi^\psi_{z,\lambda}(0) \in D(B(t+\lambda))$, and

$$k \in (I + \lambda B(t+\lambda))z - F(t+\lambda, \varphi^\psi_{z,\lambda}, z).$$

Hence $[\varphi^\psi_{z,\lambda}, z] \in D(A(t+\lambda))$, and $[\psi, k] \in (I + \lambda A(t+\lambda))[\varphi^\psi_{z,\lambda}, z] \in D(A(t+\lambda))$.

(A3)$_1$ Let $t_n \in [s, T)$, and $[\varphi_n, h_n] \in D(A(t_n))$ such that $t_n \uparrow t \in (s, T]$ and $[\varphi_n, h_n] \to [\varphi, h]$ in E_X as $n \to \infty$.
Set $\lambda_n = t - t_n$. By (2.54), we may define $[\psi_n, k_n] = J_{\lambda_n}^{A(t)}[\varphi_n, h_n]$ for n large enough. Then using (2.52) we have

$$\begin{aligned} \|[\psi_n, k_n] - [\varphi, h]\| &\leq \|[\varphi_m, h_m] - [\varphi, h]\| + \|[\psi_n, k_n] - [\varphi_m, h_m]\| \\ &\leq \|[\varphi_m, h_m] - [\varphi, h]\| + \frac{1}{1-\lambda_n\gamma}\|[\varphi_n, h_n] - [\varphi_m, h_m]\| \\ &+ \frac{\lambda_n}{1-\lambda_n\gamma}\|[a_m, b_m]\| + \frac{\lambda_n}{1-\lambda_n\gamma}L(\|[\varphi_m, h_m]\|)H(t, t_m) \end{aligned}$$

for $m \geq 1$, and $[a_m, b_m] \in A(t_m)[\varphi_m, h_m]$. Therefore,

$$\limsup_{n\to\infty} \|[\psi_n, k_n] - [\varphi, h]\| \leq 2\|[\varphi_m, h_m] - [\varphi, h]\|$$

for $m \geq 1$, which shows that $[\psi_n, k_n] \to [\varphi, h]$ as $n \to \infty$. Since $[\psi_n, k_n] \in D(A(t))$ for $n \geq 1$, it follows that $[\varphi, h] \in cl(D(A(t)))$. This completes the proof of (S1)$_1$.

According to [49], $(S2)_1$ holds.

The proof of $(S3)_1$ parallels Step 3 of the proof of Theorem 2.2 with appropriate changes. Let $[\varphi, h] \in cl(D(A(t)))$. We show that $u(\varphi, h)$ defined by (2.48) is a mild solution to the Cauchy problem

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni f(t), & s \le t \le T \\ u(s) = h, \end{cases} \tag{2.55}$$

with $f(t) = F(t, u_t, u(t))$. According to $(S1)_1$ there are sequences (t_k^n), $([\varphi_k^n, h_k^n]) \subset D(A(t_k^n))$ and $([\psi_k^n, d_k^n]) \subset E_X$ such that $s = t_0^n < t_1^n < \ldots < t_{N_{n-1}}^n < t_{N_n}^n \le T$, and

$$\frac{[\varphi_k^n, h_k^n] - [\varphi_{k-1}^n, h_{k-1}^n]}{t_{k-1}^n - t_k^n} + A(t_k^n)[\varphi_k^n, h_k^n] \ni [\psi_k^n, d_k^n] \qquad k \in \{1, \ldots, N_n\}, \tag{2.56}$$

with

$$[\varphi_0^n, h_0^n] \to [\varphi, h] \quad \text{as} \quad n \to \infty, \tag{2.57}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{N_n} \|[\psi_k^n, d_k^n]\|(t_{k-1}^n - t_k^n) = 0. \tag{2.58}$$

Moreover if $\varphi_n(t) = \begin{cases} [\varphi_0^n, h_0^n] & t = s \\ [\varphi_k^n, h_k^n] & t \in (t_{k-1}^n, t_k^n] \end{cases}$, then

$$\lim_{n\to\infty} \varphi_n(t) = U(t,s)[\varphi, h] \quad \text{uniformly over compact subsets of} \quad [s, T). \tag{2.59}$$

Now we show that u_n defined by $u_n(s) = h_0^n$, $u_n(t) = h_k^n$ for $t \in (t_{k-1}^n, t_k^n]$, is a sequence of DS–approximate solutions to (2.55). From (2.56)

$$\frac{h_k^n - h_{k-1}^n}{t_{k-1}^n - t_k^n} + \pi_2 A(t_k^n)[\varphi_k^n, h_k^n] \ni d_k^n,$$

and therefore $\frac{h_k^n - h_{k-1}^n}{t_{k-1}^n - t_k^n} + B(t_k^n)h_k^n \ni d_k^n + F(t_k^n, \varphi_k^n, h_k^n)$. By (2.57), $h_0^n \to h$ as $n \to \infty$. Moreover, by the same argument as in Step 3 of Theorem 2.2, and (2.58) we have

$$\lim_{n\to\infty} \sum_{k=1}^{N_n} \int_{t_{k-1}^n}^{t_k^n} \|d_k^n + F(t_k^n, \varphi_k^n, h_k^n) - F(\tau, \pi_1 U(\tau, s)[\varphi, h], \pi_2 U(\tau, s)[\varphi, h])\| d\tau = 0.$$

But from $(S2)_1$, $F(\tau, \pi_1 U(\tau, s)[\varphi, h], \pi_2 U(\tau, s)[\varphi, h]) = F(\tau, (u(\varphi, h))_\tau, u(\varphi, h)(\tau))$, and so u_n is a sequence of DS–approximate solutions to (2.55). Finally, according to (2.59), $u_n(.)$ converges to $\pi_2 U(., s)[\varphi, h] = u(\varphi, h)(.)$ uniformly over compact subintervals of $[s, T)$, and this completes the proof of $(S3)_1$.

Remark 2.24. Suppose that the family $F(.,.,.)$ is Lipschitz on bounded sets in the following way: given any S, $r > 0$, there exists $M(S,r) > 0$ such that for all $t \in [0,S]$, and $[\varphi,h],[\psi,k] \in \hat{E}(t) \times \hat{X}(t)$ with $\|[\varphi,h]\|, \|[\psi,k]\| \leq r$,

$$\|F(t,\varphi,h) - F(t,\psi,k)\| \leq M(S,r)\|[\varphi,h] - [\psi,k]\|. \tag{2.60}$$

Then the solutions to $(\mathrm{FDE})_1$ as asserted by Theorem 2.23 are unique amongst all mild solutions u to $(\mathrm{FDE})_1$ with $u_t \in \hat{E}(t)$ and $u(t) \in \hat{X}(t)$, $t \geq s$.

3 Existence and flow invariance of solutions to (FDE) under subtangential conditions

In Chapter 2, we studied the existence and flow invariance of solutions to (FDE) under the local range condition (B3)(ii). In this chapter we investigate existence and flow invariance of (mild) solutions to non-autonomous partial differential delay equations (FDE) under subtangential conditions. We shall start with a brief history of existence results under various subtangential conditions:

For the case of the ordinary differential delay equation $\dot{u}(t) = F(t, u_t)$,i.e. $B \equiv 0$, $\hat{X}$ a closed subset of X, and $\hat{E} = \{\varphi \in E \,|\, \varphi(0) \in \hat{X}\}$ with $E = C([-R, 0]; X)$, flow invariance of $\hat{X}$ has been shown under the following subtangential condition:

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(\varphi(0) + \lambda F(t, \varphi), \hat{X}\big) = 0 \quad \text{for all} \quad \varphi \in \hat{E}. \tag{3.1}$$

If, in addition $\hat{X}$ is convex, then the subtangential condition (3.1) for all $\varphi \in E$, with $\varphi(s) \in \hat{X}$ implies flow invariance of $\hat{X}$. (see [38, 39, 40]).

In the non–delay case of the evolution equation $\dot{u}(t) + Bu(t) \ni f(t, u(t))$, where B is an m–accretive operator,the subtangential condition

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(x + \lambda f(t, x), (I + \lambda B)(\hat{X}(t + \lambda) \cap D(B))\big) = 0 \tag{3.2}$$

for all $x \in \hat{X}(t) \cap cl(D(B))$ is a sufficient condition for flow invariance of $\hat{X}(t)$ under various assumptions on f, cf.[4]. However, for the non–delay case, the references [1, 4, 5] (also compare the references in [4]) mostly use a weaker subtangential condition in terms of the semigroup generated by $-B$ which, in particular cases, is also necessary for flow invariance. (For the semilinear case of (FDE) with $-B$ generating a C_0–semigroup of bounded linear operators, compare [41, 42].) But it is not as useful as a sufficient condition, since what is known in concrete examples is the operator B, not the semigroup it generates.

In this chapter, we provide a subtangential condition for existence and flow invariance which will extend those of the above special cases to the general case of (FDE). (For the autonomous (FDE), see [59].) Namely, we show that under certain assumptions on F and the family $B(t)$ if for all $\psi \in \hat{E}(t)$

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda} d\big(\psi(0) + \lambda F(t,\psi), (I + \lambda B(t+\lambda))(\hat{X}(t+\lambda) \cap D(B(t+\lambda)))\big) = 0,$$

then there exists a mild solution u to (FDE) such that $u(t) \in \hat{X}(t)$, $t \geq 0$. Here $\hat{X}(t)$ is a family of closed subsets of X and $\hat{E}(t)$ is a family of closed subsets in $\hat{E}_0(t)$ with $\hat{E}_0(t) = \{\varphi \in E \,|\, \varphi(0) \in cl(\hat{X}(t) \cap D(B(t)))\}$.

As in Chapter 2, our method of proof will entirely be based on the technique of transforming the original problem (FDE) in (the state space) X into a Cauchy problem in (the initial history space) E. We associate with (FDE) the family of operators $A(t)$ in E defined by

$$\begin{cases} D(A(t)) = \{\varphi \in \hat{E}_0(t) \mid \varphi' \in E, \varphi(0) \in D(B(t)), \varphi'(0) \in F(t,\varphi) - B(t)\varphi(0)\} \\ A(t)\varphi := -\varphi' \,, \varphi \in D(A(t)), \end{cases}$$

Then the main problem will be the existence and flow invariance of solutions to the corresponding non–autonomous Cauchy problem with $A(t)$. From (3.2), one would expect the following subtangential condition for the Cauchy problem $\dot{\Phi}(t) + A(t)\Phi(t) = 0$, $\Phi(s) = \psi$, i.e.,

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda} d\big(\psi, (I+\lambda A(t+\lambda))(\hat{E}(t+\lambda) \cap D(A(t+\lambda)))\big) = 0, \qquad \psi \in \hat{E}(t) \cap cl(D(A(t))).$$

However, this particular approximation of $\psi \in \hat{E}(t)$ by $(\varphi - \lambda\varphi')$ with φ simultaneously in $D(A(t+\lambda))$ and in $\hat{E}(t+\lambda)$, does not seem to be possible under the considered subtangential condition on (FDE). But following the idea of [59], we can translate it to some other sufficient condition for existence and flow invariance of mild solutions to the Cauchy problem associated with $A(t)$, the separate subtangential condition, which in the autonomous case was developed by M. Pierre. Therefore we shall need the non–autonomous version of the following result:

Theorem 3.1. [48, Theorem 2]. *Assume that $C \subset X \times X$ is accretive, and let F be a closed subset of X. Assume, moreover, that the pair (C, F) fulfills the following condition:*

$$H(0,0) \qquad \begin{cases} \forall x \in F, \forall \varepsilon > 0, \exists \lambda \in (0,\varepsilon], \exists [x_\lambda, y_\lambda] \in C, \exists u_\lambda \in F \textit{ such that} \\ \|x - (x_\lambda + \lambda y_\lambda)\| \leq \lambda\varepsilon, \quad \textit{and} \quad \|x_\lambda - u_\lambda\| \leq \lambda\varepsilon. \end{cases}$$

Then C generates a semigroup of contractions on $F \cap cl(D(C))$ which leaves this set invariant.

3.1 Existence and flow invariance of solutions to the non–autonomous Cauchy problem under the separate subtangential condition

Let X be a Banach space. Assume that the family $C(t) \subset X \times X$ of time-dependent nonlinear operators with the (possibly) time-dependent domain $D(C(t))$, and $K(t) \subset X$ are such that the following hold:

(P1) $K(t)$, $t \geq 0$ are closed subsets of X such that
(i) $\forall x \in K(t)$, $\forall \varepsilon > 0$, $\exists \lambda \in (0, \varepsilon]$, $\exists [x_\lambda, y_\lambda] \in C(t+\lambda)$, $\exists u_\lambda \in K(t+\lambda)$;

$$\|x - (x_\lambda + \lambda y_\lambda)\| \leq \lambda\varepsilon, \ \|x_\lambda - u_\lambda\| \leq \lambda\varepsilon.$$

(ii) If $t_n \uparrow t$, and $x_n \in K(t_n)$ with $x_n \to x$, then $x \in K(t)$.

(P2)(i) There exists $\gamma \in \mathbb{R}$, a bounded continuous function $h : [0, \infty) \to X$, and a bounded (on bounded subsets) function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $[x_1, y_1] \in C(t_1), [x_2, y_2] \in C(t_2), 0 \leq t_2 \leq t_1$, and $0 < \lambda < \lambda_0$,

$$(1 - \lambda\gamma)\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| + \lambda\|h(t_1) - h(t_2)\|L(\|x_2\|),$$

where $\lambda_0\gamma < 1$. Moreover, $D(C(t))$ depends on t in the following way:
(ii) If $t_n \uparrow t$, and $x_n \in D(C(t_n))$ with $x_n \to x$, then $x \in cl(D(C(t)))$.

Remark 3.2. Assume $K(t) \cap D(C(t)) \neq \emptyset$, then (P1)$(i)$ is implied by

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda} d(x, (I + \lambda C(t+\lambda))(D(C(t+\lambda)) \cap K(t+\lambda)) = 0, \quad x \in K(t). \tag{3.3}$$

We now extend [48, Theorem 2] to the non–autonomous case.

Theorem 3.3. *Suppose that the family $C(t)$ satisfies (P1) and (P2). Let $T > s \geq 0$, and $u_0 \in K(s) \cap cl(D(A(s)))$. Then the Cauchy problem*

$$\begin{cases} \dot{u}(t) + C(t)u(t) \ni 0, & s \leq t \\ u(s) = u_0, \end{cases} \tag{3.4}$$

has a unique mild solution u on $[s, T]$ such that $u(t) \in K(t) \cap cl(D(A(t)))$, $s \leq t \leq T$.

Remarks 3.4. (a) If $K(t) = cl(D(C(t)))$, the above theorem is [47, Theorem 3.6]. (Note that in [47], it is assumed that $\lim_{\lambda\to 0^+} \frac{1}{\lambda} d(x, R(I + \lambda C(t + \lambda))) = 0$, but the theorem holds as well for the following subtangential condition). Indeed, in this case (P1)(i) is equivalent to:

$$\liminf_{\lambda\to 0^+} \frac{1}{\lambda} d(x, R(I + \lambda C(t + \lambda))) = 0 \quad \text{for all} \quad x \in cl(D(C(t))). \tag{3.5}$$

(b) If $K(t) \cap D(C(t)) \neq \emptyset$, then the conclusion of Theorem 3.3 holds under the conditions (P2), (P1)(ii), and (3.3).

For the proof of Theorem 3.3, we follow the idea in [48, Theorem 2]. The following lemma will be used in the proof of Theorem 3.3.

Lemma 3.5. *Assume that conditions $P(1)$ and $P(2)$ are satisfied. Let $\varepsilon > 0$ and $u_0 \in K(s)$, then there exist sequences $\{\lambda_n\}_{n\geq 1} \in (0, \varepsilon]$, $\big\{[x_n, y_n]\big\}_{n\geq 1} \in C(s + \sum_{k=1}^{n} \lambda_k)$ and $\{u_n\}_{n\geq 1} \in K(s + \sum_{k=1}^{n} \lambda_k)$ such that:*
(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$
(ii) For all $n \geq 1$, $\|u_{n-1} - (x_n + \lambda_n y_n)\| \leq \lambda_n\varepsilon$, $\|x_n - u_n\| \leq \lambda_n\varepsilon$
Moreover, we have
(iii) $d(x_n, K(s + \sum_{k=1}^{n} \lambda_k)) \leq \lambda_n\varepsilon$ for all $n \geq 1$,
(iv) For all $n \geq 2$, $\left\|\dfrac{x_n - x_{n-1}}{\lambda_n} + y_n\right\| \leq \varepsilon_n$, where $\varepsilon_n = \varepsilon + \frac{\lambda_{n-1}}{\lambda_n}\varepsilon$.

Proof. Let $\varepsilon > 0$. We may assume $\varepsilon\gamma_0 < 1/2$, where $\gamma_0 = \max\{0, \gamma\}$. For $x \in K(t)$ the assumption $(P1)$ allows us to define

$$\begin{aligned}\lambda(x, t) = \sup\big\{\lambda \in (0, \varepsilon] \,\big|\, &\exists [x_\lambda, y_\lambda] \in C(t + \lambda),\, \exists u_\lambda \in K(t + \lambda),\\ &\|x - (x_\lambda + \lambda y_\lambda)\| \leq \lambda\varepsilon, \|x_\lambda - u_\lambda\| \leq \lambda\varepsilon\big\}.\end{aligned}$$

Let $u_0 \in K(s)$. Using the above notation there exist λ_1, with $0 < \frac{\lambda(u_0,s)}{2} \leq \lambda_1 \leq \varepsilon$, $[x_1, y_1] \in C(s + \lambda_1)$ and $u_1 \in K(s + \lambda_1)$ such that

$$\|u_0 - (x_1 + \lambda_1 y_1)\| < \lambda_1\varepsilon, \text{ and } \|x_1 - u_1\| < \lambda_1\varepsilon.$$

Next, considering $\lambda(u_1, s+\lambda_1)$, there exist λ_2, with $0 < \frac{\lambda(u_1,s+\lambda_1)}{2} \leq \lambda_2 \leq \varepsilon$, $[x_2, y_2] \in C(s + \lambda_1 + \lambda_2)$, and $u_2 \in K(s + \lambda_1 + \lambda_2)$ such that $\|u_1 - (x_2 + \lambda_2 y_2)\| < \lambda_2\varepsilon$, and $\|x_2 - u_2\| < \lambda_2\varepsilon$. Continuing this argument we define inductively sequences $\{\lambda_n\}_{n\geq 1} \in (0, \varepsilon]$, $\big\{[x_n, y_n]\big\}_{n\geq 1} \in C(s + \sum_{k=1}^{n} \lambda_k)$ and $\{u_n\}_{n\geq 1} \in K(s + \sum_{k=1}^{n} \lambda_k)$, such that

$$0 < \frac{1}{2}\lambda(u_{n-1}, s + \sum_{k=1}^{n-1} \lambda_k) \leq \lambda_n \leq \varepsilon \quad \text{for all} \quad n \geq 1,$$

and moreover $\|u_{n-1} - (x_n + \lambda_n y_n)\| \le \lambda_n \varepsilon$, and $\|x_n - u_n\| \le \lambda_n \varepsilon$. The assertions (ii), and (iii) are now obvious. For $n \ge 2$

$$\left\| \frac{x_n - x_{n-1}}{\lambda_n} + y_n \right\| \le \frac{1}{\lambda_n}\|u_{n-1} - (x_n + \lambda_n y_n)\| + \frac{1}{\lambda_n}\|x_{n-1} - u_{n-1}\| \le \varepsilon + \varepsilon \frac{\lambda_{n-1}}{\lambda_n}.$$

Therefore condition (iv) is also satisfied. It only remains to show $\sum_{k=1}^{\infty} \lambda_k = \infty$. Set $t_n = \sum_{k=1}^{n} \lambda_k$. Suppose $t_n \uparrow a < \infty$. Then $\{x_n\}_{n \ge 2}$ are bounded. Indeed, set $s_n = s + t_n$, and fix $[u, v] \in C(0)$, then using (P2)(ii) for $[u, v] \in C(0)$, and $[x_n, y_n] \in C(s_n)$, $n \ge 2$ we have

$$\begin{aligned}
&(1 - \gamma_0 \lambda_n)\|x_n - u\| \le \|x_n - u + \lambda_n(y_n - v)\| + \lambda_n L(\|u\|)\|f(s_n) - f(0)\| \\
&\le \|x_n - x_{n-1} + \lambda_n y_n\| + \|x_{n-1} - u\| + \lambda_n\|v\| + \lambda_n L(\|u\|)\|f(s_n) - f(0)\| \\
&\le \varepsilon\lambda_n + \varepsilon\lambda_{n-1} + \|x_{n-1} - u\| + \lambda_n\|v\| + \lambda_n L(\|u\|)\|f(s_n) - f(0)\|.
\end{aligned}$$

Multiplying by $\gamma_{n-1} = \prod_{k=1}^{n-1}(1 - \lambda_k \gamma_0)$ and summing from $p = 2$ to n we have

$$\begin{aligned}
\gamma_n\|x_n - u\| \;&\le\; (1 - \lambda_1\gamma_0)\|x_1 - u\| + \varepsilon\gamma_{n-1}\sum_{2}^{n}[\lambda_k + \lambda_{k-1}] \\
&+\; \gamma_{n-1}\sum_{2}^{n}\lambda_k\|v\| + \gamma_{n-1}L(\|u\|)\sum_{2}^{n}\lambda_k\|f(s_k) - f(0)\| \\
&\le\; \|x_1 - u\| + 2\varepsilon t_n + t_n\|v\| + L(\|u\|)\sum_{2}^{n}\lambda_k\|f(s_k) - f(0)\|.
\end{aligned}$$

Here we have used that $\gamma_k \le 1$. Now using the estimate

$$(1 - \xi)^{-1} \le e^{2\xi}, \quad \text{for all} \quad 0 \le \xi < 1/2 \tag{3.6}$$

we have

$$\|x_n - u\| \le e^{2a\gamma_0}\left[\lambda_1\varepsilon + \|u_0 - u\| + 2a\varepsilon + a\|v\| + 2abL(\|u\|)\right],$$

with $b = \sup\{f(\tau) \,|\, \tau \in [0, a]\}$. Thus there exists $M > 0$ depending on a, ω_0, f, u_0, u, and v such that $\|x_n\| \le M$, $n \ge 2$. As we will see later

$$\begin{aligned}
&\prod_{p=k+1}^{i}(1 - \gamma_0\lambda_p)\prod_{p=k+1}^{j}(1 - \gamma_0\lambda_p)\|x_i - x_j\| \;\le \\
&(t_i - t_j)|||C(s_k)u_k||| + \sum_{p=k+1}^{i}\lambda_p\|z_p\| + \sum_{p=k+1}^{j}\lambda_p\|z_p\| \;+ \\
&\sum_{p=k+1}^{i}\lambda_p L(M)\|f(s_p) - f(s_k)\| + \sum_{p=k+1}^{j}\lambda_p L(M)\|f(s_p) - f(s_k)\|,
\end{aligned} \tag{3.7}$$

for all $2 \le k \le j \le i$, where we set $\prod_{p=k+1}^{k}(1-\gamma_0\lambda_p) = 1$, $z_p = \frac{x_p - x_{p-1}}{\lambda_p} + y_p$, $p \ge 2$, and $|||C(s_k)u_k||| = \inf\{y \,|\, y \in C(s_k)u_k\}$.
Now using (3.6), we obtain from inequality (3.7) that

$$\begin{aligned}
&\|x_i - x_j\| \\
&\le \; e^{2\gamma_0(t_i - t_k)} e^{2\gamma_0(t_j - t_k)} \big[(t_i - t_j)|||C(t_k)u_k||| + \varepsilon[(t_i - t_k) + (t_{i-1} - t_{k-1})] \\
&+ \; \varepsilon[(t_j - t_k) + (t_{j-1} - t_{k-1})] + 2bL(M)(t_i - t_k) + 2bL(M)(t_j - t_k)\big],
\end{aligned}$$

for $2 \le k \le i \le j$. Here M, and b are as in above. Therefore

$$\limsup_{i,j\to\infty} \|x_i - x_j\| \le e^{4\gamma_0(a - t_k)} \big[2\varepsilon[(a - t_k) + (a - t_{k-1})]\big], \quad k \ge 2.$$

Hence x_j is a Cauchy sequence in X, and therefore converges.
Set $x = \lim_{n\to\infty} x_n$. Since $\|x_n - u_n\| \le \lambda_n \varepsilon$, and t_n converges, we conclude that $u_n \to x$ as $n \to \infty$. We note that $u_n \in K(s_n)$, thus (P2)(ii) implies that $x \in K(s+a)$. Then by condition $(P1)$, there exist $\mu \in (0, \varepsilon/2]$, $[x_\mu, y_\mu] \in C(s+a+\mu)$ and $u_\mu \in K(s+a+\mu)$ such that

$$\|x - (x_\mu + \mu y_\mu)\| \le \mu\varepsilon/2 \quad \text{and} \quad \|x_\mu - u_\mu\| \le \mu\varepsilon/2. \tag{3.8}$$

But $\lambda(u_n, s+t_n) \le 2\lambda_{n+1}$. Hence $\lambda(u_n, s+t_n) \to 0$, and so there exists n_0, such that for all $n \ge n_0$

$$\lambda(u_n, s+t_n) < \mu. \tag{3.9}$$

We define $h_n = a - t_n + \mu$. Then $h_n \downarrow \mu$. Choose $N_0 > n_0$ such that

$$|h_{N_0} - \mu| \; < \mu\varepsilon/4\|y_\mu\|, \text{ and } \|u_{N_0} - x\| < \mu\varepsilon/4, \tag{3.10}$$

Therefore (3.8), and (3.10) imply that

$$\begin{aligned}
\|u_{N_0} - (x_\mu + h_{N_0} y_\mu)\| \;&\le\; \|u_{N_0} - x\| + \|x - (x_\mu + \mu y_\mu)\| + \|\mu y_\mu - h_{N_0} y_\mu\| \\
&\le\; \mu\varepsilon \le h_{N_0}\varepsilon.
\end{aligned} \tag{3.11}$$

We observe that $a + \mu = t_{N_0} + h_{N_0}$. Therefore $[x_\mu, y_\mu] \in C(s + t_{N_0} + h_{N_0})$ and $u_\mu \in K(s + t_{N_0} + h_{N_0})$. Then from the definition of $\lambda(u_{N_0}, s + t_{N_0})$, (3.10), and (3.11) we conclude that $h_{N_0} \le \lambda(u_{N_0}, s + t_{N_0})$. This contradicts (3.9).

Finally we prove that the inequality (3.7) holds.
Set $\gamma_{i,j} = \prod_{p=k+1}^{i}(1-\gamma_0\lambda_p)\prod_{p=k+1}^{j}(1-\gamma_0\lambda_p)$, and $z_i = \frac{x_i - x_{i-1}}{\lambda_i} + y_i$ for all $2 \le k \le$

$j \le i$.
If $i = j$, then (3.7) is clear. Let $i > j = k$. Then since $(1-\gamma_0\lambda_i) \le (1-\gamma\lambda_i)$, and $z_i - \frac{x_i - x_{i-1}}{\lambda_i} \in C(s_i)$, inequality (P2)$(ii)$ implies that

$$\begin{aligned}(1-\gamma_0\lambda_i)\|x_i - x_k\| &\le \left\| x_i - x_k + \lambda_i \left(z_i - \frac{x_i - x_{i-1}}{\lambda_i} - |||C(s_k)x_k||| \right) \right\| \\ &+ \lambda_i L(\|u_k\|)\|f(s_i) - f(s_k)\|.\end{aligned}$$

Using that $\gamma_{i,k} = (1-\gamma_0\lambda_i)\gamma_{i-1,k}$ we have

$$\begin{aligned}\gamma_{i,k}\|x_i - x_k\| &\le \gamma_{i-1,k}\|x_{i-1} - x_k\| + \lambda_i\|z_i\| + \lambda_i|||C(s_k)x_k||| \\ &+ \lambda_i L(\|x_k\|)\|f(s_i) - f(s_k)\|,\end{aligned}$$

here we have used that $\gamma_{i-1,k} \le 1$. Now summing up over i, from $k+1$ to i we have

$$\gamma_{i,k}\|x_i - x_k\| \le$$
$$(t_i - t_k)|||C(s_k)x_k||| + \sum_{p=k+1}^{i} \lambda_p\|z_p\| + \sum_{p=k+1}^{i} \lambda_p L(M)\|f(s_p) - f(s_k)\|,$$

therefore (3.7) is true for $i > j = k$. Now let $i > j > k$, and assume (3.7) holds for $(i-1, j)$, $(i, j-1)$. If we show that the inequality is true for (i, j), then by induction (3.7) is true for all $i \ge j \ge k$.

Following the same idea as in [19, Lemma 5.1], we can show that

$$\begin{aligned}(\lambda_i + \lambda_j - \gamma_0\lambda_i\lambda_j)\|x_i - x_j\| &\le \lambda_i\|x_i - x_{j-1}\| + \lambda_j\|x_{i-1} - x_j\| \\ &+ \lambda_i\lambda_j\big(\|z_i\| + \|z_j\|\big) + \lambda_i\lambda_j L(\|M\|)\|f(s_i) - f(s_j)\|.\end{aligned} \tag{3.12}$$

Indeed, set $\sigma = \frac{\lambda_i\lambda_j}{\lambda_i + \lambda_j}$. Take $\lambda > 0$ such that $\lambda\sigma\gamma < 1$ for all $i, j \ge 1$. Since $z_p - \frac{x_p - x_{p-1}}{\lambda_p} \in C(s_p)$ for all $p \ge 2$ then $(I + \lambda\sigma C(s_i))x_i \ni x_i + \lambda\sigma\big[z_i - \frac{x_i - x_{i-1}}{\lambda_i}\big]$, and $(I + \lambda\sigma C(s_j))x_j \ni x_j + \lambda\sigma\big[z_j - \frac{x_j - x_{j-1}}{\lambda_j}\big]$. Therefore inequality (P2)(i) implies that

$$\begin{aligned}(1-\lambda\sigma\gamma_0)\|x_i - x_j\| &\le \left\| x_i + \lambda\sigma\big[z_i - \frac{x_i - x_{i-1}}{\lambda_i}\big] - x_j - \lambda\sigma\big[z_j - \frac{x_j - x_{j-1}}{\lambda_j}\big] \right\| \\ &+ \lambda\sigma L(M)\|f(s_i) - f(s_j)\|.\end{aligned}$$

Then noting that $(\lambda - \lambda\sigma\gamma_0) + (1-\lambda) = (1-\lambda\sigma\gamma_0)$, we obtain

$$(\lambda - \lambda\sigma\gamma_0)\|x_i - x_j\| + (1-\lambda)\|x_i - x_j\| \le$$
$$\left\| x_i + \lambda\sigma z_i - \lambda\frac{\lambda_j}{\lambda_i + \lambda_j}(x_i - x_j) - x_j - \lambda\sigma z_j + \lambda\frac{\lambda_i}{\lambda_i + \lambda_j}(x_j - x_i) \right\| +$$
$$\lambda\frac{\lambda_j}{\lambda_i + \lambda_j}\|x_{i-1-x_j}\| + \lambda\frac{\lambda_i}{\lambda_i + \lambda_j}\|x_i - x_{j-1}\| + \lambda\sigma L(M)\|f(s_i) - f(s_j)\|.$$

Rearranging the above inequality we conclude

$$(1-\sigma\gamma_0)\|x_i-x_j\| \le \frac{\lambda_j}{\lambda_i+\lambda_j}\big\|x_{i-1-x_j}\big\| + \frac{\lambda_i}{\lambda_i+\lambda_j}\|x_i-x_{j-1}\| +$$
$$\frac{\big\|(x_i-x_j)+\lambda(1-\lambda)^{-1}\sigma(z_i-z_j)\big\| - \|x_i-x_j\|}{\lambda(1-\lambda)^{-1}} + \sigma L(M)\|f(s_i)-f(s_j)\|.$$

Now if $\lambda \to 0$, then

$$(1-\sigma\gamma_0)\|x_i-x_j\| \le \frac{\lambda_j}{\lambda_i+\lambda_j}\big\|x_{i-1-x_j}\big\| + \frac{\lambda_i}{\lambda_i+\lambda_j}\|x_i-x_{j-1}\| +$$
$$< \sigma(z_i-z_j), x_i-x_j >_+ + \sigma L(M)\|f(s_i)-f(s_j)\|.$$

Then using that $< \sigma(z_i-z_j), x_i-x_j >_+ \le \sigma\|z_i-z_j\|$, and the definition of σ we obtain the desired inequality.

Returning to the proof of inequality (3.7), we multiply (3.12) by $\gamma_{i,j}$ to obtain

$$(\lambda_i+\lambda_j-\gamma_0\lambda_i\lambda_j)\gamma_{i,j}\|x_i-x_j\| \le$$
$$\lambda_i(1-\gamma_0\lambda_j)\gamma_{i,j-1}\|x_i-x_{j-1}\| + \lambda_j(1-\gamma_0\lambda_i)\gamma_{i-1,j}\|x_{i-1}-x_j\| +$$
$$\lambda_i\lambda_j(1-\gamma_0\lambda_i)\|z_i\| + \lambda_i\lambda_j(1-\gamma_0\lambda_j)\|z_j\| +$$
$$L(\|M\|)\big(\lambda_i\lambda_j(1-\gamma_0\lambda_i)\|f(s_i)-f(s_k)\| + \lambda_i\lambda_j(1-\gamma_0\lambda_j)\|f(s_j)-f(s_k)\|\big),$$

for all $i > j > k$. Thus by the induction assumption we have

$$(\lambda_i+\lambda_j-\gamma_0\lambda_i\lambda_j)\gamma_{i,j}\|x_i-x_j\| \le$$
$$|||C(s_k)x_k|||\big(\lambda_i(1-\gamma_0\lambda_j)(t_i-t_{j-1}) + \lambda_j(1-\gamma_0\lambda_i)(t_{i-1}-t_j)\big) +$$
$$\lambda_j(1-\gamma_0\lambda_i)\big[\sum_{p=k+1}^{i-1}\lambda_p\|z_p\| + \sum_{p=k+1}^{j}\lambda_p\|z_p\| + \lambda_i\|z_i\|\big] +$$
$$\lambda_i(1-\gamma_0\lambda_j)\big[\sum_{p=k+1}^{i}\lambda_p\|z_p\| + \sum_{p=k+1}^{j-1}\lambda_p\|z_p\| + \lambda_j\|z_j\|\big] +$$
$$\lambda_j(1-\gamma_0\lambda_i)L(M)\sum_{p=k+1}^{j}\lambda_p\|f(s_p)-f(s_k)\| +$$
$$\lambda_i(1-\gamma_0\lambda_j)L(M)\sum_{p=k+1}^{i}\lambda_p\|f(s_p)-f(s_k)\| +$$
$$\lambda_j(1-\gamma_0\lambda_i)L(M)\big[\lambda_i\|f(s_i)-f(s_k)\| + \sum_{p=k+1}^{i-1}\lambda_p\|f(s_p)-f(s_k)\|\big] +$$
$$\lambda_i(1-\gamma_0\lambda_j)L(M)\big[\lambda_j\|f(s_j)-f(s_k)\| + \sum_{p=k+1}^{j-1}\lambda_p\|f(s_p)-f(s_k)\|\big].$$

Then noting that $(t_i - t_{j-1}) = (t_i - t_j) + \lambda_j$, and $(t_{i-1} - t_j) = -\lambda_i + (t_i - t_j)$, and using

$$\lambda_i(1-\gamma_0\lambda_j) + \lambda_j(1-\gamma_0\lambda_i) \leq \lambda_i + \lambda_j - \gamma_0\lambda_i\lambda_j,$$

the above inequality implies (3.7).

We shall use the following notation.
Given $r \in [0,T]$, define $\rho : [0,T] \to \mathbb{R}^+$ by

$$\rho(r) = \sup\{\|f(t) - f(\tau)\|; t,\tau \in [0,T], |t-\tau| \ \leq r\}.$$

Obviously, ρ is bounded, nondecreasing and $\lim\limits_{r\to 0^+}\rho(r) = 0$. Moreover, ρ is upper semicontinuous on $[0,T]$ and right semicontinuous on $[0,T)$. Noting that ρ is non-decreasing on $[0,T]$, if $0 < c < \sigma \leq T$ and $0 \leq r' < \sigma - c$, the function ρ satisfy the inequality

$$\rho(r) \leq c^{-1}\rho(T)|r - r'| \ + \rho(\sigma), \quad \text{for} \quad r \in [0,T].$$

For more details see [47, page 11].
We shall also need the following lemma.

Lemma 3.6. [47, lemma 2.3] *Let $s, \hat{s} \in [0,T]$, $u_0 \in cl(D(C(s)))$, $\hat{u}_0 \in cl(D(C(\hat{s})))$, and let u_n and $\hat{u}_n$ be two DS–approximate solutions to* (3.4) *corresponding to s and u_0, respectively, $\hat{s}$ and $\hat{u}_0$. Let also (P2)(i) be satisfied and $0 \leq |\eta| \ < \sigma < T, 0 < c < \sigma - |\eta|$. Assume that $d_n, \hat{d}_m < \sigma - |\eta| \ - c$. Then for each $[\tilde{u},\tilde{v}] \in A(r)$, $r \in [0,T]$, and for $0 \leq i \leq N_n$, $0 \leq j \leq \hat{N}_m$*

$$\prod_{p=k+1}^{i}(1-\omega_0\lambda_p)\prod_{p=k+1}^{j}(1-\omega_0\hat{\lambda}_p)\big\|u_i^n - \hat{u}_j^m\big\| \ \leq$$

$$\|u_0^n - \tilde{u}\| + \|\hat{u}_0^m - \tilde{u}\| + \sum_{p=k+1}^{i}\|z_p\lambda_p\| + \sum_{p=k+1}^{j}\left\|\hat{z}_p\hat{\lambda}_p\right\| \ +$$

$$c_{i,j}(s-\hat{s})\left[\|\tilde{v}\| + M\rho(T)\right] + M(\hat{t}_j^m - \hat{s})\left[c^{-1}\rho(T)c_{i,j}(\eta) + \rho(\sigma)\right],$$

where $M = \max\{L\big(\sup\limits_{0\leq i\leq N_n}\|u_i^n\|\big), L\big(\sup\limits_{0\leq j\leq \hat{N}_m}\big\|\hat{u}_j^m\big\|\big), L(\|\tilde{u}\|)\}$, *and*

$$c_{i,j}(\eta) = \left[(t_i^n - \hat{t}_j^m - \eta)^2 + d_n(t_i^n - s) + \hat{d}_m(\hat{t}_j^m - \hat{s})\right]^{1/2}.$$

Proof of Theorem 3.3. Given $T > s$, and $u_0 \in K(s) \cap cl(D(C(s)))$. Let $\varepsilon_n \to 0^+$ such that $\varepsilon_n\gamma < 1/2$. By Lemma 3.5, for every $n \geq 1$ there exist $\{\lambda_i^n\}_{i\geq 1} \in (0,\varepsilon_n]$, $\big\{[x_i^n, y_i^n]\big\}_{i\geq 1} \in C(s + \sum_{k=1}^{i}\lambda_k^n)$, and $\{u_i^n\}_{i\geq 1} \in K(s + \sum_{k=1}^{i}\lambda_k^n)$ satisfying $(i) - (iv)$.

Set $t_0^n = s$, and $t_i^n = s + \sum_{k=1}^{i} \lambda_k^n$, $i \geq 1$. Using (i) there exists N_n such that $t_N^n \leq T+1 \leq t_{N_{n+1}}^n$.
Since $u_0 \in cl(D(C(s)))$, there exists $x_0^n \in D(C(s))$ such that $\|u_0 - x_0^n\| \leq \varepsilon_n$. If we set

$$z_i^n = \frac{x_i^n - x_{i-1}^n}{\lambda_i^n} + y_i^n \quad \text{for all} \quad i \geq 1,$$

then $z_i^n \in \frac{x_i^n - x_{i-1}^n}{\lambda_i^n} + A(t_i^n)x_i^n$. Moreover,

$$\sum_{i=1}^{N_n} \|z_i^n\|(t_i^n - t_{i-1}^n) = \sum_{i=1}^{N_n} \left\|x_i^n - x_{i-1}^n + \lambda_i^n y_i^n\right\| \quad \leq$$

$$\|x_1^n + \lambda_1^n y_1^n - u_0\| + \|x_0^n - u_0\| + \sum_{i=2}^{N_n} \left\|x_i^n - x_{i-1}^n + \lambda_i^n y_i^n\right\| \quad \leq$$

$$\varepsilon_n\big(\lambda_1 + 1 + \sum_{i=2}^{N_n} \lambda_i + \sum_{i=2}^{N_n} \lambda_{i-1}\big) \quad \leq \quad \varepsilon_n(\varepsilon_n + 1 + 2T).$$

Therefore u_n, defined by

$$u_n(t) = \begin{cases} x_0^n, & t = s \\ x_k^n, & t \in (t_{k-1}^n, t_k], \end{cases}$$

is a sequence of DS–approximate solutions to (3.4). At this point we follow the proof of [47, Theorem 3.1] to show that u_n converges uniformly to a continuous function u on $[s,T]$.

Let $t \in [s,T]$. Choose i_n, and j_m such that $t \in (t_{i_n-1}^n, t_{i_n}^n] \cap (t_{j_m-1}^m, t_{j_m}^m]$. From the lines of proof of Lemma 3.5, we can read that x_i^n $n \in \mathbb{N}$, and $0 \leq i \leq N_n$ are bounded. Take $\tilde{x} \in D(C(s))$, such that $x_0^n \to \tilde{x}$. Then using (3.6), and Lemma 3.6 for discrete scheme $\{x_n, z_n\}$, and with $\eta = 0$ we have

$$\left\|x_{j_m}^m - x_{i_n}^n\right\| \leq e^{4\gamma_0(T-s)}\big(\|x_0^m - \tilde{x}\| + \|x_0^n - \tilde{x}\| + M_1(T-s)\rho(\sigma)\big), \tag{3.13}$$

with $M_1 = \max\{L\big(\sup_{0\leq i\leq N_n} \|x_i^n\|\big), L(\|\tilde{x}\|)\}$, and $\sigma > 0$. Since $u_n(t) = x_{i_n}^n$, and $u_m(t) = x_{j_m}^m$, and $\rho(\sigma) \to 0$, as $\sigma \to 0$, inequality (3.13) implies that

$$\lim_{m,n\to\infty} \|u_n(t) - u_m(t)\| = 0,$$

and therefore u_n converges uniformly to a function u over $[s,T]$.
It can also be shown that any other DS–approximate solution $\hat{u}_n$ corresponding to s and u_0 is also convergent to u. Moreover, u is uniformly continuous on $[s,T]$. For

the proof see ([47, Theorem 3.1]).
We next show that for $t \in (s,T]$, $u(t) \in K(t) \cap cl(D(C(t)))$. Let $t \in (s,T]$. We may find a subsequence $t^n_{k(n)-1}$ such that $t^n_{k(n)-1} \uparrow t$. Thus $u_n(t^n_{k(n)-1}) = x^n_{k(n)-1}$ is a sequence in $D(C(t^n_{k(n)-1}))$ which converges to $u(t)$. Therefore by (P2)(ii), $u(t) \in cl(D(C(t)))$. Moreover, by Lemma 3.5 $d(x^n_{k(n)-1}, u^n_{k(n)-1}) \leq \varepsilon_n^2$. Therefore $u^n_{k(n)-1} \to u(t)$ as $n \to \infty$. (P1)(ii) now implies that $u(t) \in K(t)$, and this completes the proof.

Remark 3.7. The following can be read from [47, Theorem 3.2 and Theorem 3.5]:
1. Let u be the mild solution to (3.4) as in Theorem 3.3. Then u is the unique integral solution to (3.4).
2. The family $\{U(t,s) \,|\, U(t,s) : K(s) \cap cl(D(C(s))) \to K(t) \cap cl(D(C(t)))\}$ of operators associated with $C(t)$ via $U(t,s)u_0 = u(t)$, $0 \leq s \leq t \leq T$ is an evolution operator of type γ.

3.2 Existence of mild solutions to (FDE)

3.2.A The initial history space. Given $I = (-\infty, 0]$ or $I = [-R, 0]$ for some $R > 0$, the initial history space E is assumed to be a Banach space of continuous functions $\varphi : I \to X$ satisfying (E.1), (E.2), and (E.3) with (E.1)(b), and (E.1)(c), respectively replaced by:

(E.1)(b′) For all $x \in X$, $\bar{x} \in E$, where $\bar{x}(s) \equiv x$, $s \in I$, and there exists $C_E \geq 1$ such that $\|\bar{x}\| \leq C_E \|x\|$ for all $x \in X$.

(E.1)(c′) For φ, $(\varphi_n)_n$ in E, if $\|\varphi_n - \varphi\| \to 0$, then $\|\varphi_n(s) - \varphi(s)\| \to 0$ for all $s \in I$, and moreover

$$\int_\alpha^\beta \varphi_n(s)ds \to \int_\alpha^\beta \varphi(s)ds \quad \text{for all} \quad \alpha, \beta \in I,\, \alpha < \beta.$$

The initial history spaces considered in Remark 2.1, all satisfy axioms (E.1)(b′), and (E.1)(c′).

3.2.B Assumptions. Given an initial history space E as above, we consider the following assumptions:

(B1) $(B(t))_{0 \leq t}$ is a family of operators $B(t) \subset X \times X$ such that there exist $\alpha \in \mathbb{R}$, a continuous function $f : \mathbb{R}^+ \to X$, and a nondecreasing bounded function (on bounded sets) $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $[x_i, y_i] \in B(t_i), i \in \{1, 2\}$, and

$0 \le t_2 \le t_1$,

$$(1-\lambda\alpha)\|x_1 - x_2\| \le \|x_1 - x_2 + \lambda(y_1 - y_2)\| + \lambda\|f(t_1) - f(t_2)\|L_1(\|x_2\|), \tag{3.14}$$

for all $\lambda > 0$ with $\lambda\alpha < 1$.

(B2) $\hat{X}(t) \subset X$, $t \ge 0$ are closed subsets of X such that $\hat{X}(t) \cap D(B(t)) \neq \emptyset$, with the following t–dependence :
If $t_n \uparrow t \in (0,\infty)$, and $x_n \in \hat{X}(t_n) \cap D(B(t_n))$ such that $x_n \to x$, then $x \in cl(\hat{X}(t) \cap D(B(t)))$.

(B3) $\hat{E}_0(t) = \{\varphi \in E \,|\, \varphi(0) \in cl(\hat{X}(t) \cap D(B(t)))\}$, $t \ge 0$, and $\hat{E}(t)$ are closed subsets of $\hat{E}_0(t)$ such that
(i) For $x \in \hat{X}(t+\lambda) \cap D(B(t+\lambda))$, $\psi \in \hat{E}(t)$, and $\lambda > 0$, sufficiently small, if $\varphi^{\psi}_{x,\lambda} \in E$ is the solution to $\varphi - \lambda\varphi' = \psi$, $\varphi(0) = x$, then $\varphi^{\psi}_{x,\lambda} \in \hat{E}(t+\lambda)$.
(ii) If $t_n \uparrow t \in (0,\infty)$, $\varphi_n \in \hat{E}(t_n)$, and $\varphi_n \to \varphi$ in E, then $\varphi \in \hat{E}(t)$.

(B4) The operator $F : \bigcup\limits_{t\in[0,\infty)} \{t\} \times \hat{E}_0(t) \to X$ is such that
(i) F is continuous and bounded on bounded sets.
(ii) There exists $M > 0$ such that for $\varphi_i \in \hat{E}_0(t_i)$, $i \in \{1,2\}$, $0 \le t_2 \le t_1$, with $\|\varphi_1 - \varphi_2\| = \|\varphi_1(0) - \varphi_2(0)\|$,

$$\begin{aligned} < F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1(0) - \varphi_2(0) >_+ &\le M\|\varphi_1 - \varphi_2\| \\ &+ \|g(t_1) - g(t_2)\|L_2(\|\varphi_2\|). \end{aligned}$$

where $g : \mathbb{R}^+ \to X$ is a continuous function and $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded (on bounded sets) function.
(iii) there exists $M' > 0$ such that, if $\varphi_1, \varphi_2 \in \hat{E}_0(t)$, $t \ge 0$ with $\varphi_1(0) = \varphi_2(0)$, then

$$\|F(t,\varphi_1) - F(t,\varphi_2)\| \le M'\|\varphi_1 - \varphi_2\|.$$

Lemma 3.8. *The inequality* (3.14) *is equivalent to*

$$-\alpha\|x_1 - x_2\| \le < y_1 - y_2, x_1 - x_2 >_+ + \|f(t_1) - f(t_2)\|L_1(\|x_2\|), \tag{3.15}$$

for all $[x_i, y_i] \in B(t_i), i \in \{1,2\}$, $0 \le t_2 \le t_1$.

Proof. The proof of the above lemma is an easy computation based on relation (1.2).

We now formulate the main result of this chapter.

Theorem 3.9. *Given the assumptions* (B1)–(B4)*, assume that for all* $\psi \in \hat{E}(t)$*,*

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(\psi(0) + \lambda F(t,\psi), (I + \lambda B(t+\lambda))(\hat{X}(t+\lambda) \cap D(B(t+\lambda)))\big) = 0. \quad (3.16)$$

Then we have
(i) for all $\psi \in \hat{E}(s)$*,* $s \geq 0$*, there exists a global mild solution* u_ψ *to*

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t), & s \leq t \\ u_s = \psi. \end{cases}$$

such that $(u_\psi)_t \in \hat{E}(t)$*,* $t \geq s$*. In particular* $u_\psi(t) \in \hat{X}(t)$ *for all* $t \geq s$*.*
(ii) If, in addition $F(.,.)$ *is Lipschitz continuous on bounded sets (see* (3.17)*), or given any* S*,* $r > 0$*, there exists* $M'(S,r) > 0$ *such that for all* $t \in [0,S]$*, and* $\varphi, \psi \in \hat{E}(t)$ *with* $\|\varphi\|, \|\psi\| \leq r$*,*

$$< \varphi(0) - \psi(0), F(t,\varphi) - F(t,\psi) >_+ \leq M'(S,r)\|\varphi - \psi\|, \quad (3.17)$$

then for any $\psi \in \hat{E}(s)$*, the mild solution* u_ψ *as in (i) is unique amongst all mild solutions* u *to (FDE) with the property that* $u_t \in \hat{E}(t)$ *for all* $t \geq s$*.*

Remark 3.10. Let $\psi_1, \psi_2 \in \hat{E}(s)$, and let u_{ψ_1} and u_{ψ_2} be the corresponding mild solutions as in (i), then according to Remark 2.7, u_{ψ_i}, $i \in \{1,2\}$ is an integral solution to (FDE), and by Lemma 2.29 we have:

$$\begin{gathered} e^{-\alpha t}\|u_{\psi_1}(t) - u_{\psi_2}(t)\| - e^{-\alpha s}\|u_{\psi_1}(s) - u_{\psi_2}(s)\| \leq \\ \int_s^t e^{-\alpha\tau} < F(\tau, (u_{\psi_1})_\tau) - F(\tau, (u_{\psi_2})_\tau), u_{\psi_1}(\tau) - u_{\psi_2}(\tau) >_+ d\tau, \end{gathered} \quad (3.18)$$

for all $0 \leq s \leq t$. Moreover,

$$\|(u_{\psi_1})_t - (u_{\psi_2})_t\| \leq e^{\omega t}\|\psi_1 - \psi_2\| \quad \text{for all} \quad t \geq 0, \quad (3.19)$$

where $\omega = max\{0, \alpha + M\}$,

Remark 3.11. If in addition to the assumptions (B1) and (B2),

$$\hat{X}(t) \cap cl(D(B(t))) \subset R(I + \lambda B(t)), \quad \text{and} \quad J_\lambda^{B(t)}[\hat{X}(t) \cap cl(D(B(t)))] \subset \hat{X}(t)$$

for all $\lambda > 0$, small enough, then $cl(\hat{X}(t) \cap D(B(t))) = \hat{X}(t) \cap cl(D(B(t)))$. Thus in Theorem 3.9, the set $\hat{E}_0(t)$ can be chosen as $\hat{E}_0(t) = \{\varphi \in E \,|\, \varphi(0) \in \hat{X}(t) \cap cl(D(B(t)))\}$. We note that this latter set is the largest possible set of initial histories, for which we can expect existence, and flow invariance of mild solutions to (FDE). Indeed, for the existence of mild solutions, we need $u_\psi(0) = \psi(0) \in cl(D(B(s)))$, and for the invariance, it is required that $u_\psi(0) = \psi(0) \in \hat{X}(s)$.

Corollary 3.12. *Given the assumptions (B1), (B2), and (B4), assume that for all $\psi \in \hat{E}_0(t)$,*

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(\psi(0) + \lambda F(t, \psi), (I + \lambda B(t+\lambda))(\hat{X}(t+\lambda) \cap D(B(t+\lambda)))\big) = 0. \quad (3.20)$$

Then for all $\psi \in \hat{E}_0(s)$ there exists a global mild solutions u_ψ to (FDE) such that $u_\psi(t) \in \hat{X}(t)$ for all $t \geq s$.

Proof. We note that if $\hat{E}(t) = \hat{E}_0(t)$, then (B3)($i$) is automatically fulfilled. Moreover, condition (B3)(ii) follows from (B2). The above corollary, is then Theorem 3.9 with $\hat{E}(t) = \hat{E}_0(t)$.

In the results for the ordinary delay case of $B = 0$, the set $\hat{E}$ has been taken as $\hat{E} = \{\varphi \in \hat{E}_0 \,|\, \varphi(s) \in \hat{X} \quad \text{for all} \quad s \in I\}$, where $\hat{X} \subset X$ is closed and convex, and $\hat{E}_0 = \{\varphi \in E \,|\, \varphi(0) \in cl(\hat{X} \cap D(B))\}$. Following this idea, we take up a special case of Theorem 3.9, with $\hat{X}(t), t \geq 0$ closed and convex subsets of X, and

$$\hat{E}(t) = \{\varphi \in \hat{E}_0(t) \,|\, \varphi(s) \in \hat{X}(t) \quad \text{for all} \quad s \in I\}.$$

Note that, in this case, if the sets $\hat{X}(t)$ are nondecreasing (i.e. $\hat{X}(r) \subseteq \hat{X}(t), r \leq t$), then since $\hat{X}(t)$ are closed and convex, condition (B3)(i) is fulfilled. If in addition we assume the following;

$$\text{If } t_n \uparrow t \in (0, \infty),\ x_n \in \hat{X}(t_n), \quad \text{and } x_n \to x \in X, \quad \text{then } x \in \hat{X}(t), \qquad (3.21)$$

then from (B2), it follows that (B3)(ii) is also satisfied. This together with Theorem 3.9 leads to the following result.

Theorem 3.13. *Let $\hat{X}(t), t \geq 0$ be closed and convex subsets of X, and $\hat{E}(t) = \{\varphi \in \hat{E}_0(t) \,|\, \varphi(s) \in \hat{X}(t) \quad \text{for all} \quad s \in I\}$. Assume in addition that $\hat{X}(t)$ are nondecreasing, and the conditions (B1), (B2), (B4), and* (3.21) *are satisfied. If for all $\psi \in \hat{E}(t)$;*

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(\psi(0) + \lambda F(t, \psi), (I + \lambda B(t+\lambda))(\hat{X}(t+\lambda) \cap D(B(t+\lambda)))\big) = 0,$$

then the conclusions of Theorem 3.9 hold.

Following the idea in [4, Lemma 4.2], we can separate the subtangential condition (3.16) of Theorem 3.9 in the following way (compare [59]);

Lemma 3.14. *Under the assumptions (B1)-(B4), if for all* $t \geq 0$*, and* $\lambda > 0$ *small enough;*

$$\hat{X}(t) \subset R(I + \lambda B(t)), \qquad J_\lambda^{B(t)} \hat{X}(t) \subset \hat{X}(t), \tag{3.22}$$

and moreover

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}\, d\big(\psi(0) + \lambda F(t, \psi), \hat{X}(t+\lambda)\big) = 0, \tag{3.23}$$

then the conclusions of Theorem 3.9 hold.

Proof. We show that (3.22) and (3.23) imply that (3.16) holds. By (3.23), there exist $\lambda_n \to 0^+$, and $y_n \in \hat{X}(t + \lambda_n)$ such that

$$\frac{1}{\lambda_n} \|\psi(0) + \lambda_n F(t, \psi) - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

But by (3.22), $x_n = J_{\lambda_n}^{B(t+\lambda_n)} y_n \in \hat{X}(t + \lambda_n) \cap D(B(t + \lambda_n)$, and therefore

$$\frac{1}{\lambda_n} d\big(\psi(0) + \lambda_n F(t, \psi), (I + \lambda_n B(t + \lambda_n)(x_n)\big) \leq$$
$$\frac{1}{\lambda_n} \|\psi(0) + \lambda_n F(t, \psi) - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

By the above theorem, the flow invariance problem is reduced to two independent conditions: one on $\hat{X}(.)$, $\hat{E}(.)$, and F, and one on the resolvents of the family $B(t)$. In particular, if $B(t)$, $t \geq 0$ are α–m–accretive, and $J_\lambda^{B(t)}$, $t \geq 0$ leave $\hat{X}(t)$ invariant, then the subtangential condition (3.23) implies flow invariance of $\hat{X}(.)$ and $\hat{E}(.)$. Actually, there is a prominent subclass of accretive operators for which this reduction of the subtangential condition 3.16 to (3.23) can be achieved for natural choices of the family $\hat{X}(t)$ (see Section 3.5).

3.3 Proof of Theorem 3.9.

Our method of proof will be based on the technique of transforming the original problem (FDE) in (the state space) X to a Cauchy problem in (the initial history space) E.

We associate with (FDE) the family of operators $A(t)$ in E defined by

$$\begin{cases} D(A(t)) = \{\varphi \in \hat{E}_0(t) \mid \varphi' \in E, \varphi(0) \in D(B(t)), \varphi'(0) \in F(t, \varphi) - B(t)\varphi(0)\} \\ A(t)\varphi := -\varphi' \,, \varphi \in D(A(t)), \end{cases} \tag{3.24}$$

Considering the operator $A(t)$, we shall prove that if F satisfies (B4)(iii), then $cl(D(A(t))) = \hat{E}_0(t)$ for all $t \geq s$. So under the assumptions of Theorem 3.9,

$$\hat{E}(t) \cap cl(D(A(t))) = \hat{E}(t).$$

Following [59], for the proof of Theorem 3.9, we prove the following assertions:

(S1) For every $\psi \in \hat{E}(s)$, there exists a mild solution φ_ψ to the Cauchy problem

$$\begin{cases} \dot{\Phi}(t) + A(t)\Phi(t) = 0, & s \leq t \\ \Phi(s) = \psi. \end{cases} \tag{3.25}$$

The evolution operator $U(t,s)$, $t \geq s$ generated by $A(t)$ via $U(t,s)\psi := \varphi_\psi(t)$ is such that $U(t,s)\hat{E}(s) \subset \hat{E}(t)$.

(S2) If $\psi \in \hat{E}(s)$, and $u_\psi : (I+s) \cup [s,\infty) \to X$ is defined by

$$u_\psi(t) = \begin{cases} \psi(t-s), & I \ni t-s \leq 0 \\ (U(t,s)\psi)(0), & t \geq s \end{cases}$$

then $U(t,s)\psi = (u_\psi)_t$, for all $s \leq t$.

(S3) For $\psi \in \hat{E}(t)$, the function u_ψ of (S2) is a global mild solution to (FDE).

As in [59], we shall need the following auxiliary results for the proof of Theorem 3.9.

Lemma 3.15. *Let C be the operator in E defined by*

$$\begin{cases} D(C) = \{\varphi \in E \mid \varphi' \in E, \varphi'(0) = 0\} \\ C\varphi := -\varphi', \varphi \in D(C). \end{cases}$$

Then C is a (linear) m–accretive operator with dense domain. In particular

$$\lim_{\lambda \to 0} J_\lambda^C \rho = \rho \quad \textit{for all} \quad \rho \in E.$$

To determine $cl(D(A(t)))$, for $A(t)$ defined in (3.24), we shall need the next lemma.

Lemma 3.16. *Let $\rho \in E$, $x \in D(B(t)) \cap \hat{X}(t)$, $y \in B(t)x$, $t \geq 0$, and $\lambda > 0$ such that $\lambda M' C_E < 1$. Let $\hat{E}_{0,x}(t) = \{\varphi \in \hat{E}_0(t) \,|\, \varphi(0) = x\}$. Then the operator $T : \hat{E}_{0,x}(t) \to \hat{E}_{0,x}(t)$ defined by*

$$\varphi \mapsto \left\{ s \mapsto e^{s/\lambda}x + \frac{e^{s/\lambda}}{\lambda} \int_s^0 e^{-\xi/\lambda}\big[\rho(\xi) + (x + \lambda y - (\rho(0) + \lambda F(t,\varphi)))\big] d\xi \right\}$$

has a unique fixed point $\tilde{\varphi} \in \hat{E}_{0,x}(t)$ such that $\tilde{\varphi} \in D(A(t))$.

Proof. $T\varphi$ is the solution to $\begin{cases} \Phi - \lambda\Phi' = \rho + x + \lambda y - (\rho(0) + \lambda F(t,\varphi)), \\ \Phi(0) = x. \end{cases}$

It is easy to see that T is a contraction. Actually (E.2), (B4)(iii), and (E.1)(b′) imply that

$$\|T\varphi_1 - T\varphi_2\| \leq \lambda \left\| \overline{F(t,\varphi_1) - F(t,\varphi_2)} \right\| \leq$$
$$\lambda C_E \|F(t,\varphi_1) - F(t,\varphi_2)\| \leq \lambda M' C_E \|\varphi_1 - \varphi_2\|$$

for all $\varphi_i \in E_{0,x}(t)$, $i \in \{1,2\}$. By the assumption on λ, we conclude that T is a contraction, and thus there exists $\tilde{\varphi} \in \hat{E_{0,x}}(t)$ such that $T\tilde{\varphi} = \tilde{\varphi}$. So $\tilde{\varphi}(0) = x \in D(B(t))$, and $\tilde{\varphi} - \lambda\tilde{\varphi}' = \rho + x + \lambda y - (\rho(0) + \lambda F(t,\tilde{\varphi}))$. This implies that $\tilde{\varphi} \in D(A(t))$. Indeed,

$$\tilde{\varphi}'(0) = \frac{x + \lambda y - \lambda F(t,\tilde{\varphi}) - \tilde{\varphi}(0)}{-\lambda} \in F(t,\tilde{\varphi}) - B(t)\tilde{\varphi}(0).$$

Proposition 3.17. *Let $A(t)$ as in* (3.24)*. If F satisfies* (B4)(iii)*, then*

$$cl(D(A(t))) = \hat{E}_0(t).$$

Proof. Obviously $cl(D(A(t))) \subset \hat{E}_0(t)$. Let $\varphi \in \hat{E}_0(t)$. Given $\varepsilon > 0$. There exists $a \in D(B(t)) \cap \hat{X}(t)$ such that $\|a - \varphi(0)\| < \varepsilon$. Pick $b \in B(t)a$. Choose $\lambda > 0$ sufficiently small such that $\lambda M' C_E < 1$. Then applying Lemma 3.16 with $\rho = \varphi$, $x = a$, and the above λ there exists $\tilde{\varphi} \in D(A(t))$ such that

$$\tilde{\varphi}(0) = a, \quad \text{and} \quad \tilde{\varphi} - \lambda\tilde{\varphi} = \varphi + a + \lambda b - (\varphi(0) + \lambda F(t,\tilde{\varphi})).$$

Set $\theta = \varphi + a + \lambda b - (\varphi(0) + \lambda F(t,\tilde{\varphi}))$. Then

$$\begin{aligned} \|\tilde{\varphi} - \varphi\| = \left\| J_\lambda^{A(t)}\theta - \varphi \right\| &\leq \left\| J_\lambda^{A(t)}\theta - \theta \right\| + \left\| \overline{a - \varphi(0)} \right\| + \lambda \left\| \overline{b - F(t,\tilde{\varphi})} \right\| \\ &\leq \left\| J_\lambda^{A(t)}\theta - \theta \right\| + C_E\|a - \varphi(0)\| + \lambda C_E \|b - F(t,\tilde{\varphi})\|. \end{aligned} \tag{3.26}$$

Lemma 2.5(b), and (E.2) imply that;

$$\begin{aligned} \left\| J_\lambda^{A(t)}\theta - \theta \right\| &\leq \|J_{0,\lambda}(\theta - \theta(0)) - (\theta - \theta(0))\| + \left\| (J_\lambda^{A(t)}\theta)(0) - \theta(0) \right\| \\ &= \|J_{0,\lambda}(\varphi - \varphi(0)) - (\varphi - \varphi(0))\| + \lambda\|b - F(t,\tilde{\varphi})\|. \end{aligned} \tag{3.27}$$

Since $(\varphi + a - \varphi(0))(0) = \tilde{\varphi}(0)$, using (B4)($iii$) we may write

$$\begin{aligned} \|F(t,\tilde{\varphi})\| &\leq \|F(t,\varphi + a - \varphi(0))\| + M'\|\varphi + a - \varphi(0) - \tilde{\varphi}\| \\ &\leq \|F(t,\varphi + a - \varphi(0))\| + M' C_E \|a - \varphi(0)\| + M'\|\varphi - \tilde{\varphi}\|. \end{aligned} \tag{3.28}$$

Combining (3.26), (3.27), and (3.28) we have

$$(1-2\lambda M')\|\varphi-\tilde\varphi\| \le \|J_{0,\lambda}(\varphi-\varphi(0))-(\varphi-\varphi(0))\| + C_E(1+2M')\|a-\varphi(0)\| \\ +\lambda(1+C_E)\|b\| + \lambda(1+C_E)\|F(t,\varphi+a-\varphi(0))\|. \quad (3.29)$$

Since $\varphi-\varphi(0)\in E_0$, from Lemma 2.5(a) it follows that

$$\lim_{\lambda\to 0}\|J_{0,\lambda}(\varphi-\varphi(0))-(\varphi-\varphi(0))\| = 0.$$

Thus for $\lambda>0$ small enough, inequality (3.29) implies that φ is ε–close to $\tilde\varphi$ in norm, and therefore $\varphi\in cl(D(A(t))$.

Remark 3.18. Consider the above operators $A(t)$ with $D(A(t))\subset \hat E_0(t) := \{\varphi\in E\,|\,\varphi(0)\in \hat X(t)\cap cl(D(B(t)))\}$. If (B4)$(iii)$ is satisfied, then

$$cl(D(A(t))) = \{\varphi\in E\,|\,\varphi(0)\in cl(\hat X(t)\cap D(B(t))).\}$$

(The proof follows by the same argument as in Proposition 3.17.)

Proof of Theorem 3.9. We proceed by proving assertions (S1)-(S3).

Proof of (S1). In this part of proof we shall translate our assumptions on $B(t)$ to those in Theorem 3.3 for the family $A(t)$ defined by (3.24). The assertion (S1) then can be read from Theorem 3.3.

1. Let $\omega=\max\{0,M+\alpha\}$. Given $\lambda>0$ with $\lambda\omega<1$. Let $\varphi_1\in D(A(t_1))$ and $\varphi_2\in D(A(t_2))$, $s\le t_2\le t_1$. As $\psi=\varphi_1-\varphi_2$ solves the equation $\psi-\lambda\psi' = (\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')$ with $\psi(0)=\varphi_1(0)-\varphi_2(0)$, by (E.2) we have

$$\|\varphi_1-\varphi_2\| \le \max\{\|\varphi_1(0)-\varphi_2(0)\|, \|(\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')\|\}.$$

In case $\|\varphi_1(0)-\varphi_2(0)\|\le\|(\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')\|$, we have the desired inequality for the family $A(t)$. Otherwise, (E.1)(a) implies that

$$\|\varphi_1-\varphi_2\| = \|\varphi_1(0)-\varphi_2(0)\|, \quad (3.30)$$

and therefore according to (B4)(ii);

$$< F(t_1,\varphi_1)-F(t_2,\varphi_2), \varphi_1(0)-\varphi_2(0)>_+ \;\le\; M\|\varphi_1-\varphi_2\| \\ +\; \|g(t_1)-g(t_2)\|L_2(\|\varphi_2\|). \quad (3.31)$$

Moreover, $[\varphi_i(0), F(t_i,\varphi_i)-\varphi_i'(0)] \in B(t_i)$ for $i\in\{1,2\}$. Then using Lemma 3.8 we have

$$\begin{aligned}-\alpha\|\varphi_1-\varphi_2\| &\le\ < -\varphi_1'(0)+\varphi_2'(0)+F(t_1,\varphi_1)-F(t_2,\varphi_2), \varphi_1(0)-\varphi_2(0) >_+\\ &+\ \|f(t_1)-f(t_2)\|L_1(\|\varphi_2(0)\|).\end{aligned}$$

Therefore, invoking (3.31) in the above inequality, and using that L_1 is nondecreasing we obtain

$$\begin{aligned}-\alpha\|\varphi_1-\varphi_2\| &\le\ < -\varphi_1'(0)+\varphi_2'(0), \varphi_1(0)-\varphi_2(0) >_+ +M\|\varphi_1-\varphi_2\|\\ &+\ \|g(t_1)-g(t_2)\|L_2(\|\varphi_2\|)+\|f(t_1)-f(t_2)\|L_1(\|\varphi_2\|). \qquad (3.32)\end{aligned}$$

Using the definition of $< .,. >_+$, (3.30), and (E.1)(a) we have

$$< -\varphi_1'(0)+\varphi_2'(0), \varphi_1(0)-\varphi_2(0) >_+ \le \frac{\|(\varphi_1-\varphi_2+\lambda(-\varphi_1'+\varphi_2')\|-\|\varphi_1-\varphi_2\|}{\lambda}$$

for all $\lambda>0$. This together with inequality (3.32) imply that

$$\begin{aligned}(1-\lambda(M+\alpha))\|\varphi_1-\varphi_2\| &\le \|(\varphi_1-\varphi_2+\lambda(-\varphi_1'+\varphi_2')\| \qquad (3.33)\\ &+\ \lambda\big(\|f(t_1)-f(t_2)\|+\|g(t_1)-g(t_2)\|\big)L(\|\varphi_2\|),\end{aligned}$$

where $L=L_1+L_2$.
Comparing (3.33) to (P2)(i), we have two control functions f and g. However the proof of Theorem 3.3 remains true. Indeed, it is only the modulus of continuity of the control function which has a role in the proof.
For the sake of simplicity, we define

$$H(t_1,t_2) := \|f(t_1)-f(t_2)\|+\|g(t_1)-g(t_2)\|. \qquad (3.34)$$

2. Let $\varphi_n\in D(A(t_n))$, and $t_n\in[s,\infty)$ such that $t_n\uparrow t\in(s,\infty)$ and $\varphi_n\to\varphi$ in E as $n\to\infty$. Then $\varphi_n(0)\in D(B(t_n))\cap\hat X(t_n)$. Therefore according to (B2) $\varphi(0)\in cl(\hat X(t)\cap D(B(t)))$. By Proposition 3.17, we conclude that $\varphi\in cl(D(A(t)))$. Thus (P2)(ii) also holds.

3. In this part we show that $(A(t),\hat E(t))$ satisfies (P1)(i). Take $t\ge 0$, and $\psi\in\hat E(t)$. Given $\varepsilon>0$. We claim that there exists λ_0 sufficiently small, with $0<\lambda_0<1/2M'C_E$, $x_{\lambda_0}\in\hat X(t+\lambda_0)\cap D(B(t+\lambda_0))$, and $y_{\lambda_0}\in B(t+\lambda_0)x_{\lambda_0}$ such that;

$$\lambda_0^{-1}\Big\|\psi(0)+\lambda_0F(t+\lambda_0,\varphi^{\psi}_{\lambda_0,x_{\lambda_0}})-(x_{\lambda_0}+\lambda_0y_{\lambda_0})\Big\|\le\varepsilon/2C_E. \qquad (3.35)$$

Note that $\varphi^{\psi}_{\lambda_0,x_{\lambda_0}}(0) = x_{\lambda_0} \in \hat{X}(t+\lambda_0) \cap D(B(t+\lambda_0))$, and $\psi \in \hat{E}(t)$. By (B3)(i), it follows that $\varphi^{\psi}_{\lambda_0,x_{\lambda_0}} \in \hat{E}(t+\lambda_0)$, and so $F(t+\lambda_0, \varphi^{\psi}_{\lambda_0,x_{\lambda_0}})$ is defined.

For the moment, let us assume (3.35) holds (we shall prove it later). We then look for elements in $D(A(t+\lambda_0))$, and $\hat{E}(t+\lambda_0)$ such that (P1)(i) is satisfied. In order to find the corresponding elements in $D(A(t+\lambda_0))$, we shall apply Lemma 3.16 to $\rho = \psi$, $\lambda = \lambda_0$, $t = t+\lambda_0$, $x = x_{\lambda_0}$, and $y = y_{\lambda_0}$ to get $\varphi_{\lambda_0} \in D(A(t+\lambda_0))$ such that

$$\varphi_{\lambda_0} - \lambda_0 \varphi'_{\lambda_0} = \psi + x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \varphi_{\lambda_0})), \quad \varphi_{\lambda_0}(0) = x_{\lambda_0}. \tag{3.36}$$

Set $\psi_{\lambda_0} = \varphi^{\psi}_{\lambda_0,x_{\lambda_0}}$. Then $\psi_{\lambda_0} \in \hat{E}(t+\lambda_0)$. Since $\psi_{\lambda_0}(0) = \varphi_{\lambda_0}(0)$, from (E.2) and (E.1)(b′), it follows that

$$\big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\| \le C_E \big\|x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \varphi_{\lambda_0}))\big\|.$$

Moreover, (B4)(iii) implies that

$$\begin{aligned} \big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\| &\le C_E \|x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \psi_{\lambda_0}))\| \\ &+ \lambda_0 M' C_E \big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\|. \end{aligned}$$

Rearranging the above inequality, and using (3.35) we obtain

$$\begin{gathered} 1/2\big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\| \le (1 - \lambda_0 M' C_E)\big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\| \le \\ C_E \|x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \psi_{\lambda_0}))\| \le \lambda_0 \varepsilon/2, \end{gathered} \tag{3.37}$$

and so $\big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\| \le \lambda_0 \varepsilon$. Next we show that $\big\|\psi - (\varphi_{\lambda_0} + \lambda \varphi'_{\lambda_0})\big\| \le \lambda_0 \varepsilon$. According to (3.36) and (E.1)(b′) we have

$$\big\|\psi - (\varphi_{\lambda_0} + \lambda \varphi'_{\lambda_0})\big\| \le C_E \big\|x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \varphi_{\lambda_0}))\big\|.$$

Thus applying (B4)(iii) to the above inequality we obtain

$$\begin{aligned} \big\|\psi - (\varphi_{\lambda_0} + \lambda \varphi'_{\lambda_0})\big\| &\le C_E \|x_{\lambda_0} + \lambda_0 y_{\lambda_0} - (\psi(0) + \lambda_0 F(t+\lambda_0, \psi_{\lambda_0}))\| \\ &+ \lambda_0 M' C_E \big\|\psi_{\lambda_0} - \varphi_{\lambda_0}\big\|. \end{aligned}$$

Then from (3.35), (3.37), and the condition on λ_0 it follows that

$$\big\|\psi - (\varphi_{\lambda_0} + \lambda \varphi'_{\lambda_0})\big\| \le \lambda_0 \varepsilon.$$

Therefore, if (3.35) is satisfied, then $(A(t), \hat{E}(t))$ fulfills (P1)(i).

To complete this part of proof we need to show that (3.35) holds. According to

(3.16) there exist $\varepsilon_n \to 0^+$ with $\varepsilon_n \leq \varepsilon$, $\lambda_n \to 0^+$, $x_n \in D(B(t+\lambda_n)) \cap \hat{X}(t+\lambda_n)$, and $y_n \in B(t+\lambda_n)x_n$ such that

$$\lambda_n{}^{-1}\|\psi(0) + \lambda_n F(t,\psi) - (x_n + \lambda_n y_n)\| \leq \varepsilon_n. \tag{3.38}$$

To show (3.35), We shall prove that

$$(\lambda_n)^{-1}\left\|\psi(0) + \lambda_n F(t+\lambda_n, \varphi^{\psi}_{\lambda_n,x_n}) - (x_n + \lambda_n y_n)\right\| \to 0. \tag{3.39}$$

We note that $\varphi^{\psi}_{\lambda_n,x_n}(0) = x_n \in \hat{X}(t+\lambda_n) \cap D(B(t+\lambda_n))$, and $\psi \in \hat{E}(t)$. Then (B2) implies that $\varphi^{\psi}_{\lambda_n,x_n} \in \hat{E}(t+\lambda_n)$, so we may write;

$$\lambda_n{}^{-1}\left\|\psi(0) + \lambda_n F(t+\lambda_n, \varphi^{\psi}_{\lambda_n,x_n}) - (x_n + \lambda_n y_n)\right\| \leq$$
$$\lambda_n{}^{-1}\|\psi(0) + \lambda_n F(t,\psi) - (x_n + \lambda_n y_n)\| + \left\|F(t,\psi) - F(t+\lambda_n, \varphi^{\psi}_{\lambda_n,x_n})\right\|.$$

Thus by the above inequality and (3.38), we only need to show that

$$\lim_{n\to\infty}\left\|F(t,\psi) - F(t+\lambda_n, \varphi^{\psi}_{\lambda_n,x_n})\right\| \to 0. \tag{3.40}$$

We first show that $\left\|\psi - \varphi^{\psi}_{\lambda_n,\psi(0)}\right\| \to 0$. Actually, according to Lemma 3.15, $\lim\limits_{\lambda\to 0^+} J^C_\lambda \psi = \psi$. But $\varphi^{\psi}_{\lambda_n,\psi(0)} \in D(C)$, and so $\varphi^{\psi}_{\lambda_n,\psi(0)} = J^C_{\lambda_n}\psi$. Hence

$$\lim_{n\to\infty} \varphi^{\psi}_{\lambda_n,\psi(0)} = \psi. \tag{3.41}$$

Therefore, if we show $\left\|\varphi^{\psi}_{\lambda_n,\psi(0)} - \varphi^{\psi}_{\lambda_n,x_n}\right\| \to 0$, then by continuity of F, inequality (3.40) holds, and so (3.39) is satisfied. According to (E.2),

$$\left\|\varphi^{\psi}_{\lambda_n,\psi(0)} - \varphi^{\psi}_{\lambda_n,x_n}\right\| \leq \|\psi(0) - x_n\|.$$

We shall prove that $\|\psi(0) - x_n\| \to 0$.
Using that $\psi \in \hat{E}(t) \subset \hat{E}_0(t)$, we may choose $[a_m, b_m] \in B(t)$ such that $\|a_m - \psi(0)\| \to 0$, as $m \to \infty$. Then

$$\begin{aligned}\|\psi(0) - x_n\| &\leq \|\psi(0) - a_m\| + \|a_m - x_n\| \qquad (3.42)\\ &= \|\psi(0) - a_m\| + \left\|J^{B(t)}_{\lambda_n}(a_m + \lambda_n b_m) - J^{B(t+\lambda_n)}_{\lambda_n}(x_n + \lambda_n y_n)\right\|.\end{aligned}$$

Condition (B1) now implies that;

$$\begin{aligned}&(1-\lambda_n\alpha)\left\|J^{B(t)}_{\lambda_n}(a_m + \lambda_n b_m) - J^{B(t+\lambda_n)}_{\lambda_n}(x_n + \lambda_n y_n)\right\| \qquad (3.43)\\ &\leq \|(a_m + \lambda_n b_m) - (x_n + \lambda_n y_n)\| + L(\|a_m + \lambda_n b_m\|)\|f(t) - f(t+\lambda_n)\|\\ &\leq \|a_m - \psi(0)\| + \|\psi(0) - (x_n + \lambda_n y_n)\| + \lambda_n\|b_m\|\\ &+ L(\|a_m\| + \|b_m\|)\|f(t) - f(t+\lambda_n)\|\end{aligned}$$

By (3.38), we have

$$\|\psi(0) - (x_n + \lambda_n y_n)\| \leq \lambda_n \varepsilon_n + \lambda_n \|F(t, \psi)\|.$$

This together with (3.42), and (3.43) imply that

$$\limsup_{n\to\infty} \|\psi(0) - x_n\| \leq 2\|a_m - \psi(0)\| \quad \text{for all} \quad m \in \mathbb{N}.$$

Thus $\lim_{n\to\infty} \|x_n - \psi(0)\| = 0$, and so $\lim_{n\to\infty} \left\|\varphi^{\psi}_{\lambda_n,\psi(0)} - \varphi^{\psi}_{\lambda_n,x_n}\right\| = 0$. Then from (B4)$(i)$ and (3.41), we conclude that

$$\left\|F(t, \psi) - F(t + \lambda_n, \varphi^{\psi}_{\lambda_n,x_n})\right\| \to 0,$$

and consequently (3.39) holds.
Therefore according to Theorem 3.3 the Cauchy problem

$$\begin{cases} \dot{\varphi}(t) + A(t)\varphi(t) = 0, \quad s \leq t \\ \varphi(s) = \psi, \end{cases} \tag{3.44}$$

has a global mild solution φ_ψ such that $\varphi_\psi(t) \in \hat{E}(t)$, $t \geq s$.

Proof of (S2). Following the lines of proof in [49, Theorem 3.1], we conclude that $U(t, s)$ acts as a translation.

Remarks 3.19. 1. We note that to be able to follow the proof in [49] we shall additionally need property (E.1)(b′) on E. **2.** By our assumptions on E, in the proof of [49, Theorem 3.1], we only obtain $\psi(\theta, t)$ is separately continuous on $I \times [0, T]$. However, the proof of [49, Lemma 2.1] still holds.

Proof of (S3). This part of proof follows by the same argument as Step 3 in the proof of Theorem 2.2.

Using Remark 3.10, proof of assertion (ii) follows by similar argument as in the proof of Proposition 2.10. This completes the proof of Theorem 3.9.

3.4 Relation between the local range condition and the subtangential condition

In Chapter 2, we investigated the existence and flow invariance of solutions to (FDE) under the following local range condition: for $x \in \hat{X}(t + \lambda)$, $\psi \in \hat{E}(t)$, $\lambda > 0$ with $\lambda\omega < 1$, where $\omega := \max\{0, M + \alpha\}$,

$$\left[\psi(0) + \lambda F(t + \lambda, \varphi^{\psi}_{x,\lambda})\right] \in (I + \lambda B(t + \lambda))(D(B(t + \lambda)) \cap \hat{X}(t + \lambda)). \tag{3.45}$$

In the following proposition, we shall study the relationship between the above local range condition and the subtangential condition (3.16).

Proposition 3.20. *Under the conditions (B1)–(B4), with (B4)(ii) replaced by condition (B.2)(ii) in Chapter 2, assume that for all $\psi \in \hat{E}(t)$, $x \in \hat{X}(t+\lambda)$, and $\lambda > 0$ with $\lambda\omega < 1$, condition* (3.45) *is satisfied. Then the subtangential condition* (3.16) *holds.*

Proof. Let $\psi \in \hat{E}(t)$. Take $\lambda_n \to 0^+$ with $\lambda_n\omega < 1$. From the proof of Proposition 2.3, we can read that $\hat{E}(t) \subset R(I + \lambda A(t+\lambda))$, $\lambda\omega < 1$. Therefore there exist $\varphi_n \in D(A(t+\lambda_n))$ such that $(I + \lambda_n A(t+\lambda_n))\varphi_n = \psi$. Then $x_n := \varphi_n(0) \in \hat{X}(t+\lambda_n) \cap D(B(t+\lambda_n))$, and so by (3.45), there exist $y_n \in (D(B(t+\lambda_n))) \cap \hat{X}(t+\lambda_n))$, and $z_n \in B(t+\lambda_n)y_n$ such that

$$\psi(0) + \lambda_n F(t+\lambda_n, \varphi^{\psi}_{x_n,\lambda_n}) = y_n + \lambda_n z_n.$$

Then

$$\frac{1}{\lambda_n}\|\psi(0) + \lambda_n F(t,\psi) - (y_n + \lambda_n z_n)\| \le \left\|F(t,\psi) - F(t+\lambda_n, \varphi^{\psi}_{x_n,\lambda_n})\right\|.$$

If we show $\varphi^{\psi}_{x_n,\lambda_n} \to \psi$, as $n \to \infty$, then continuity of F implies that (3.16) holds. We note that $\left\|\varphi^{\psi}_{x_n,\lambda_n} - \psi\right\| = \left\|J^{A(t+\lambda_n)}_{\lambda_n}\psi - \psi\right\|$, so using the same argument as in the proof of Proposition 2.13, it is enough to show

$$\left\|(J^{A(t+\lambda_n)}_{\lambda_n}\psi)(0) - \psi(0)\right\| \to 0 \quad \text{as} \quad n \to \infty.$$

If we set $\theta = \lambda_n F(t+\lambda_n, J^{A(t+\lambda_n)}_{\lambda_n}\psi) + \psi(0)$, then $(J^{A(t+\lambda_n)}_{\lambda_n}\psi)(0) = J^{B(t+\lambda_n)}_{\lambda_n}\theta$. Moreover, since $\psi \in \hat{E}(t)$, we may choose $[a_m, b_m] \in B(t)$ such that $\|a_m - \psi(0)\| \to 0$, as $m \to \infty$. Therefore,

$$\left\|(J^{A(t+\lambda_n)}_{\lambda_n}\psi)(0) - \psi(0)\right\| \le \left\|J^{B(t+\lambda_n)}_{\lambda_n}\theta - J^{B(t)}_{\lambda_n}[a_m + \lambda_n b_m]\right\| + \|\psi(0) - a_m\|. \quad (3.46)$$

Condition (B1) now implies that;

$$(1-\lambda_n\alpha)\left\|J^{B(t+\lambda_n)}_{\lambda_n}\theta - J^{B(t)}_{\lambda_n}(a_m + \lambda_n b_m)\right\| \le \|\psi(0) - a_m\| +$$
$$\lambda_n\left\|F(t+\lambda_n, J^{A(t+\lambda_n)}_{\lambda_n}\psi)\right\| + \lambda_n\|b_m\| + L(\|a_m + \lambda_n b_m\|)\|f(t) - f(t+\lambda_n)\|.$$

By (3.33), we can show that $J^{A(t+\lambda_n)}_{\lambda_n}\psi$, $n \ge 1$ is bounded, and so by inequality (3.46), we have

$$\limsup_{n\to\infty}\left\|(J^{A(t+\lambda_n)}_{\lambda_n}\psi)(0) - \psi(0)\right\| \le 2\|a_m - \psi(0)\| \quad \text{for all} \quad m \in \mathbb{N}.$$

Thus $\lim_{n\to\infty}\left\|(J_{\lambda_n}^{A(t+\lambda_n)}\psi)(0)-\psi(0)\right\|=0$, and therefore $\lim_{n\to\infty}\left\|\varphi^{\psi}_{x_n,\lambda_n}-\psi\right\|=0$, and so by continuity of F, we conclude that

$$\frac{1}{\lambda_n}\|\psi(0)+\lambda_n F(t,\psi)-(y_n+\lambda_n z_n)\|\to 0,$$

and so (3.16) is fulfilled.

3.5 Population models

The existence results in this chapter are particularly convenient for applications to population models, considered in Chapter 2, in the L^1–context. In order to apply our flow–invariance and asymptotic results on models of this type, we need the following auxiliary results.

Notation: Let $N : I\times X\to\mathbb{R}$ be convex on X, X a Banach space. Then for fixed $t\in I$, $\lambda\to N(t,x+\lambda y)$ is convex on $\mathbb{R}$ for $x,y\in X$. So we may set

$$\partial N(t,x,y):=\lim_{\lambda\to 0^+}\frac{N(t,x+\lambda y)-N(t,x)}{\lambda}=\inf_{\lambda>0}\frac{N(t,x+\lambda y)-N(t,x)}{\lambda},\quad x,y\in X.$$

Lemma 3.21. *Let $B(t)\subset X\times X$, $t\geq 0$ be a family of α–accretive operators in a Banach space X, and $C(t)\subset X$, $t\geq 0$ closed convex subsets of X. Let $N(t,x):=$ dist$(x,C(t)), x\in X$. Moreover, assume that, for $\lambda>0$ small enough, $R(I+\lambda B(t))\supset C(t)$, and $J_\lambda^{B(t)}C(t)\subset C(t)$. Then for any $t\geq 0$, $N(t,.)$ is convex and continuous, and $\partial N(t,x,y)\geq -\alpha N(t,x)$ for all $[x,y]\in B(t)$.*

Proof. The continuity and convexity properties of $N(t,.)$ on X for fixed $t\geq 0$ are obvious. Let $[x,y]\in B(t)$, $t\geq 0$, and $\lambda>0$ such that $\lambda\alpha<1$. Let $(c_n)_n$ be a sequence in $C(t)$ such that $\|x+\lambda y-c_n\|\to N(t,x+\lambda y)$. By the assumptions $J_\lambda^{B(t)}c_n\in C(t)$, and so $N(t,x)\leq\left\|x-J_\lambda^{B(t)}c_n\right\|\leq\frac{1}{1-\lambda\alpha}\|x+\lambda y-c_n\|$. Thus $(1-\lambda\alpha)N(t,x)\leq N(t,x+\lambda y)$ for $\lambda>0$ small enough. Rearranging, and letting $\lambda\to 0^+$, completes the proof .

Following the idea of [3, Theorem 19.8], we state the next theorem:

Theorem 3.22. *Let $(B(t))_{t\geq 0}$ be a family of α–m–accretive operators in X, satisfying (B1). Let I be an interval in $\mathbb{R}^+$, $M : I\times X\to(-\infty,\infty]$ be lower semicontinuous, and $N : I\times X\to\mathbb{R}$ be continuous, and for any $t\in I$, $N(t,.)$ be convex on X. Assume for any $t\in I$, and $[x,y]\in B(t)$*

$$M(t,x)\leq\limsup_{h\to 0^+}\frac{N(t-h,x)-N(t,x)}{h}+\partial N(t,x,y).\tag{3.47}$$

If $f \in L^1_{loc}(I;X)$, and u is a mild solution to $\dot{u}(t) + B(t)u(t) \ni f(t)$ on I, then the following holds:

$$N(t,u(t)) + \int_s^t M(\tau,u(\tau))d\tau \le N(s,u(s)) + \int_s^t \partial N(\tau,u(\tau,f(\tau))d\tau. \tag{3.48}$$

For the proof we shall need the following lemma, compare [3, Theorem 19.3].

Lemma 3.23. *Let $(B(t))_{t\ge 0}$ be a family of α–m–accretive operators in X, satisfying (B1). Let I be an interval in $\mathbb{R}$, and $M, N : I \times D \to (-\infty,\infty]$ be lower semicontinuous, where $D = cl(D(B(t)))$, $t \ge 0$. Moreover, assume there exists a function $\varepsilon : (0,\lambda_0] \times I \times D \to (-\infty,\infty] \to \mathbb{R}$ such that for all $(t,x) \in I \times D$, and $0 < \lambda \le \lambda_0$ with $t+\lambda \in I$;*

$$N(t+\lambda, J_\lambda(t+\lambda)x) + \lambda M(t+\lambda, J_\lambda(t+\lambda)x) \le N(t,x) + \lambda\varepsilon(\lambda,t,x), \tag{3.49}$$

and

$$\lim_{(\lambda,t,x)\to(0,t_0,x_0)} \varepsilon(\lambda,t,x) = 0 \quad \textit{for all} \quad (t_0,x_0) \in I \times D. \tag{3.50}$$

Then for every $(t,x) \in I \times D$ and $\lambda > 0$ with $t+\lambda \in I$;

$$N(t+\lambda, U(t+\lambda,t)x) + \int_t^{t+\lambda} M(\tau,U(\tau,t)x)d\tau \le N(t,x), \tag{3.51}$$

where U is the evolution operator generated by $B(t)$, and $J_\lambda(t+\lambda) = J_\lambda^{B(t+\lambda)}$.

Remark 3.24. It can be shown that the α–m–accretivity of $B(t)$, and (B1) imply that $cl(D(B(t))) = D$ is necessarily independent of t (See [47, Remark 4.2]), so by Theorem 1.7 the family $B(t)$ generates an evolution operator. Moreover, according to [47, Corollary 3.2], the evolution operator associated to $B(t)$ is given by

$$U(t,s)x = \lim_{n\to\infty} \prod_{j=1}^n J_{\frac{t-s}{n}}(s + j\frac{t-s}{n})x \quad \text{for all} \quad x \in D.$$

Proof of Lemma 3.23. We shall follow the idea of proof in [3, Theorem 19.3]. By induction on (3.49), we can show that for all $t \in I$, $0 < h \le \lambda_0$, and $n \in \mathbb{N}$ with $t + nh \in I$;

$$N(t+nh, \prod_{i=1}^n J_h(t+ih)x) + h\sum_{i=1}^n M(t+ih, \prod_{k=1}^i J_h(t+kh)x) \tag{3.52}$$
$$\le N(t,x) + h\sum_{i=1}^n \varepsilon(h, t+(i-1)h, \prod_{k=1}^{i-1} J_h(t+kh)x).$$

Let $t \in I$, and $\lambda > 0$ such that $t+\lambda \in I$. Then applying (3.52) with $h = \lambda/n$, for n sufficiently large such that $\lambda/n \leq \lambda_0$ we have

$$\begin{aligned}
&N(t+\lambda, \prod_{i=1}^{n} J_{\lambda/n}(t+i\frac{\lambda}{n})x) + \lambda/n \sum_{i=1}^{n} M(t+i\frac{\lambda}{n}, \prod_{k=1}^{i} J_{\lambda/n}(t+k\frac{\lambda}{n})x) \\
&\leq N(t,x) + \lambda/n \sum_{i=1}^{n} \varepsilon(\lambda/n, t+(i-1)\frac{\lambda}{n}, \prod_{k=1}^{i-1} J_{\lambda/n}(t+k\frac{\lambda}{n})x).
\end{aligned}$$

Then

$$\begin{aligned}
&N(t+\lambda, \prod_{i=1}^{n} J_{\lambda/n}(t+i\frac{\lambda}{n})x) + \int_t^{t+\lambda} M(\theta_n(\tau), u_n(\tau))d\tau \\
&\quad \leq N(t,x) + \int_t^{t+\lambda} \varepsilon(\lambda/n, \theta_n(\tau-\frac{\lambda}{n}), u_n(\tau-\frac{\lambda}{n}))d\tau, \qquad (3.53)
\end{aligned}$$

where $\theta_n(\tau) = t + i\frac{\lambda}{n}$, and $u_n(\tau) = \prod_{k=1}^{i} J_{\lambda/n}(t+k\frac{\lambda}{n})x$ for $\tau \in (t+(i-1)\frac{\lambda}{n}, t+i\frac{\lambda}{n}]$, $i \geq 0$. Since $\theta_n(\tau) \to \tau$, and $u_n(\tau) \to U(\tau,t)x$ uniformly on $[t, t+\lambda]$, then lower semicontinuity of M and N imply that

$$\begin{aligned}
&N(t+\lambda, U(t+\lambda,t)x) + \int_t^{t+\lambda} M(\tau, U(\tau,t)x)d\tau \\
\leq \quad &\liminf_{n\to\infty} \left[N(t+\lambda, \prod_{i=1}^{n} J_{\lambda/n}(t+i\frac{\lambda}{n})x) + \int_t^{t+\lambda} M(\theta_n(\tau), u_n(\tau))d\tau\right],
\end{aligned}$$

where we have used that (3.50) holds uniformly for all (t_0, x_0) in any fixed compact set. This together with (3.53) completes the proof.

Proof of Theorem 3.22. As in the proof of [3, Theorem 19.8], we start by assuming that $f = 0$, and N_t, the partial derivative of N with respect to t, exists and is continuous on $I \times X$. Let $t \in I$, $x \in cl(D(B(t)))$, and $\lambda > 0$, with $t + \lambda \in I$. Then applying (3.47) with $t = t+\lambda$, and $[J_\lambda(t+\lambda)x, B_\lambda(t+\lambda)x] \in B(t+\lambda)$ we have

$$M(t+\lambda, J_\lambda(t+\lambda)x) \leq -N_t(t+\lambda, J_\lambda(t+\lambda)x) + \partial N(t+\lambda, J_\lambda(t+\lambda)x, B_\lambda(t+\lambda)x),$$

and so by definition of ∂N we obtain

$$\lambda M(t+\lambda, J_\lambda(t+\lambda)x) \leq -\lambda N_t(t+\lambda, J_\lambda(t+\lambda)x) + N(t+\lambda, x) - N(t+\lambda, J_\lambda(t+\lambda)x).$$

Therefore

$$\begin{aligned}
&N(t+\lambda, J_\lambda(t+\lambda)x) + \lambda M(t+\lambda, J_\lambda(t+\lambda)x) \\
&\leq N(t,x) + [N(t+\lambda,x) - N(t,x)] - \lambda N_t(t+\lambda, J_\lambda(t+\lambda)x) \\
&= N(t,x) + \lambda\varepsilon(\lambda, t, x),
\end{aligned}$$

where $\varepsilon(\lambda, t, x) = \frac{1}{\lambda}\int_t^{t+\lambda}[N_t(\tau, x) - N_t(t+\lambda, J_\lambda(t+\lambda)x)]d\tau$. Then since $x \in cl(D(B(t))) = D$, by condition (B1) on $B(t)$ it follows that $\|J_\lambda(t+\lambda)x) - x\| \to 0$, and so continuity of N_t, implies that ε satisfies (3.50). Therefore by Lemma 3.22, (3.51) holds. Take $u_0 \in D$, $t, s \in I$ with $s < t$. We note that $U(.,0)u_0$ is the mild solution to $\dot{u}(\tau) + B(\tau)u(\tau) \ni 0$, $u(0) = u_0$. Then applying (3.51) with $t = s$, $\lambda = t - s$, and $x = U(s,0)u_0$, we have

$$N(t, U(t,0)u_0) + \int_s^t M(\tau, U(\tau,0)u_0)d\tau \leq N(s, U(s,0)u_0),$$

which is assertion (3.48) for $f = 0$.
For the general case with f nonzero and N without regularity restriction, the same discussion as in [3, Theorem 19.8] implies that (3.48) holds.

Conventions
1. Ω is an open subset of $\mathbb{R}^N$.
2. For $\beta \geq 0$, we set $[0,\beta] := \{x \in L^1(\Omega) \mid 0 \leq x(\omega) \leq \beta \quad \text{a.e.} \quad \omega \in \Omega\}$.
3. For $x \in L^1(\Omega)$, and $r \in \mathbb{R}$, $(x < r) := \{\omega \in \Omega \,|\, x(\omega) < r\}$. In the analogous way we also define $(x = r)$, and $(x > r)$.

Lemma 3.25. *Let $\beta \geq 0$. Then the following hold:*

(i) $$d(x,[0,\beta]) = -\int_{(x<0)} x + \int_{(x>\beta)} (x-\beta) \quad \textit{for all} \quad x \in L^1(\Omega). \tag{3.54}$$

(ii) If $N(x) = d(x,[0,\beta])$, $x \in L^1(\Omega)$, then for all $x, y \in L^1(\Omega)$

$$\partial N(x,y) = \int_{(x=0)} y^- + \int_{(x=\beta)} y^+ - \int_{(x<0)} y + \int_{(x>\beta)} y. \tag{3.55}$$

Proposition 3.26. *Let $B(t) \subset L^1(\Omega) \times L^1(\Omega)$ be a family of m-completely accretive operators satisfying condition (B1) with $0 \in B(t)0$ for all $t \geq 0$. Consider the delay equation*

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni a(t)u(t)\left[1 - b(t)u(t) - \int_{-1}^0 u(t+r(s))d\eta_t(s)\right] & t \geq 0 \\ u_{|[-R,0]} = \varphi, \end{cases} \tag{3.56}$$

with initial–history space $E := C([-R,0];X)$. Assume $\beta : \mathbb{R}^+ \to (0,\infty)$ is a bounded differentiable function such that β' is also bounded and satisfies the following property:

$$a(t)\beta(t) - a(t)b(t)\beta^2(t) \leq \beta'(t) \quad \textit{for all} \quad t \geq 0. \tag{3.57}$$

Let $\varphi \in C([-R,0];X)$ *with* $\varphi(s) \geq 0$ *for all* $s \in [-R,0]$*, and* $\varphi(0) \leq \beta(0)$ *a.e. on* Ω*. If in addition* $\varphi(0) \in cl(D(B(0)))$*, then the following hold:*
(i) There exists a unique global mild solution u_φ *to (3.56) such that* $(u_\varphi)_t \geq 0$ *for all* $t \geq 0$*, and* $u_\varphi(t) \in [0,\beta(t)]$ *for all* $t \geq 0$*.*
(ii) Assume Ω *has finite Lebesgue measure, and that the control function* f *in (B1) is bounded on* $\mathbb{R}^+$*. Then* u_φ *has relatively compact range if there exists* $r_0 > 0$*, such that* $J_\lambda^{B(r_0)}$*,* $\lambda > 0$ *transforms* L^∞*–bounded subsets of* $L^1(\Omega)$ *into relatively compact sets in* $L^1(\Omega)$*, and either (a)* u_φ *is uniformly continuous, or (b) the evolution operator generated by* $B(t)$ *is* T*–periodic for some* $T > 0$*, and for any* L^∞*–bounded subset* $K \subset cl(D(B(s)))$*, the family of functions* $\{U_B(t,s)x \,|\, x \in K\}$*,* $t \geq s \geq 0$*, is equicontinuous at any* $t \geq 0$ *.*
(iii) If the family $(B(t))_{t\geq 0} = B$ *is independent of* t*, then*

$$\|u_\varphi(t)\| \leq e^{\int_0^t a(\tau)d\tau}\|\varphi(0)\| \quad \textit{for all} \quad t \geq 0,$$

and if, in addition, $B + \alpha I$ *is accretive for some* $\alpha < 0$*, then*

$$\|u_\varphi(t)\| \leq e^{\int_0^t (\alpha + a(\tau))d\tau}\|\varphi(0)\| \quad \textit{for all} \quad t > s.$$

In particular, if $\alpha < 0$*, and* $a_1 < -\alpha$*, then all these solutions tend exponentially to the zero function.*

Remark 3.27. Under the assumptions of Proposition 3.26, let $\beta : \mathbb{R}^+ \to (0,\infty)$ be a bounded nondecreasing differentiable function such that $\beta(t) \geq \frac{1}{b(t)}$, $t \geq 0$, and β' is bounded. Then (3.57) is satisfied, and so the conclusions of Proposition 3.26 hold. If in addition we assume b is non–increasing and differentiable, and b' is bounded, then in particular we may take $\beta(t) = \frac{1}{b(t)}$.

Following the idea of [59, Proposition 5.1(c)], in case, $\beta(t) = \beta$ is independent of t, and $\beta \geq \sup\{\frac{1}{b(t)} \,|\, t \geq 0\}$, we can improve the above result.

Definition 3.28. For a function $u : \mathbb{R}^+ \to X$, X a Banach space, $\omega(u) := \{x \in X \,|\, u(t_n) \to x \quad \text{for some sequence} \quad t_n \to \infty\}$,respectively $\omega_w(u) := \{x \in X \,|\, u(t_n) \rightharpoonup x \quad \text{weakly for some sequence} \quad t_n \to \infty\}$, denote the (norm) omega-limit set, respectively the weak omega-limit set of u.

Proposition 3.29. *Let* $\varphi \in C([-R,0];X)$ *with* $\varphi(s) \geq 0$ *for all* $s \in [-R,0]$ *a.e. on* Ω*, and* $\varphi(0) \in cl(D(B(0))) \cap L^\infty(\Omega)$*. Then for* $\beta_0 = \sup\{\frac{1}{b(t)} \,|\, t \geq 0\}$*, the following assertions hold:*

(i) $\lim_{t\to\infty} d\left(u_\varphi(t), [0,\beta_0]\right) = 0$.
(ii) If u_φ *has relatively compact range, then* $\omega(u_\varphi) \subset [0,\beta_0]$.
(iii) In case Ω *has finite Lebesgue–measure,* u_φ *has weakly relatively compact range, and* $\omega_w(u_\varphi) \subset [0,\beta_0]$.

Proof of Proposition 3.26. Let $0 < a_0 \le a(t) \le a_1$, and $0 < b_0 \le b(t) \le b_1$. Take $\beta : \mathbb{R}^+ \to (0,\infty)$, satisfying (3.57) with $\beta = \sup\{\beta(t) \,|\, t \in \mathbb{R}^+\}$, and let $\hat{X}(t) = [0,\beta(t)]$. Since the family $B(t)$ is m–completely accretive, according to Remark 3.11, $\hat{X}(t) \cap cl(D(B(t))) = cl(\hat{X}(t) \cap D(B(t)))$. Therefore we may set

$$\hat{E}_0(t) = \{\varphi \in C([-r,0];X) \,|\, \varphi(0) \in \hat{X}(t) \cap cl(D(B(t)))\},$$

and $\hat{E}(t) = (\hat{E}_0(t))^+ := \{\varphi \in \hat{E}_0(t) \,|\, \varphi(s) \ge 0,\ s \in [-R,0]\}$. Then the equation (3.56) has the form of (FDE) with $F(t,.) : \hat{E}_0(t) \to X$, defined by $F(t,\varphi) = a(t)\varphi(0)\left[1 - b(t)\varphi(0) - G(t,\varphi)\right]$, where $G(t,\varphi) = \int_{-1}^{0} \varphi(r(s))d\eta_t(s)$. For the moment, consider this equation with the right hand side F^+, say (FDE)$^+$, with

$$F^+(t,\varphi) = a(t)\varphi(0)\left[1 - b(t)\varphi(0) - (G(t,\varphi))^+\right] \quad \text{for} \quad \varphi \in \hat{E}_0(t).$$

We then show that for (FDE)$^+$, conditions (B1)–(B4) are fulfilled in our setting. Indeed, (B1) holds by our assumptions. To check (B2), we first note that under the assumptions of Proposition 3.26, $cl(D(B(t))) = D$ is independent of t. Now let $t_n \uparrow t$, and $x_n \in \hat{X}(t_n) \cap cl(D(B(t_n)))$ with $x_n \to x$ in L^1. Then $x \in D$. Moreover, there exists a subsequence, say again, x_n such that $x_n(\omega) \to x(\omega)$ for almost every $\omega \in \Omega$. Therefore by continuity of $\beta(.)$, we conclude that $x(\omega) \in [0,\beta(t)]$ for almost every $\omega \in \Omega$. Thus $x \in \hat{X}(t) \cap D$, and so by the above discussion (B2) holds. Condition (B3) is a matter of routine checking. (Note that here, because of the special form of $\hat{E}(t)$, we do not need any restriction on $\hat{X}(t)$ or more precisely on $\beta(t)$ to get (B3)(i).)
Concerning (B4), we first take $\varphi_1, \varphi_2 \in \hat{E}_0(t)$, $t \ge 0$ with $\varphi_1(0) = \varphi_2(0)$. Then

$$\left\|F^+(t,\varphi_1) - F^+(t,\varphi_2)\right\| = a(t)\left\|\varphi_1(0)(G^+(t,\varphi_1) - G^+(t,\varphi_2))\right\| \le a(t)\beta(t)\|\eta_t\|\|\varphi_1 - \varphi_2\|,$$

so (B4)(iii) holds with $M' = a_1\beta$. To show (B4)(ii), let $\varphi \in \hat{E}(t)$ and $\psi \in \hat{E(s)}, 0 \le s \le t$. Take $x^* \in J(\varphi(0) - \psi(0))$, i.e., $x^* \in L^\infty(\Omega)$, $\|x^*\| \le 1$, and $< x^*, \varphi(0) - \psi(0) >= \|\varphi(0) - \psi(0)\|$. Then

$$< x^*, F^+(t,\varphi) - F^+(s,\psi) >= a(t)\int x^*(\varphi(0) - \psi(0)) + [a(t) - a(s)]\int x^*\psi(0)$$
$$-a(t)b(t)\int x^*(\varphi(0) - \psi(0))(\varphi(0) + \psi(0)) + [a(s)b(s) - a(t)b(t)]\int x^*\psi(0)\psi(0)$$

$$-a(t)\int x^*\varphi(0)(G^+(t,\varphi)-G^+(t,\psi))+[a(s)-a(t)]\int x^*\varphi(0)G^+(t,\psi)$$
$$-a(s)\int x^*(\varphi(0)-\psi(0))G^+(t,\psi)-a(s)\int x^*\psi(0)(-G^+(t,\psi)+G^+(s,\psi)),$$

and so

$$\begin{aligned}< x^*, F^+(t,\varphi)-F^+(s,\psi) >\leq a(t)\|\varphi-\psi\| + |a(t)-a(s)|\ \|\psi\| \\ +|a(s)b(s)-a(t)b(t)|\ \beta(s)\|\psi\| + a(t)\beta(t)\|\varphi-\psi\|\|\eta_t\| \\ +|a(t)-a(s)|\ \beta(t)\|\eta_t\|\|\psi\| + a(s)\beta(s)\|\psi\|\|\eta_t-\eta_s\|.\end{aligned}$$

Therefore

$$\begin{aligned}< x^*, F^+(t,\varphi)-F^+(s,\psi) >\leq (a_1+a_1\beta)\|\varphi-\psi\| + \\ L\big(\|\psi\|\big)\,[|a(t)-a(s)|\ +|\eta_t-\eta_s|\ +|a(t)b(t)-a(s)b(s)|\ ]\,,\end{aligned}$$

where $L:\mathbb{R}^+\to\mathbb{R}^+$ is defined by $L(x)=x+\beta(2+a_1)x$. Then by Proposition 1.2,

$$< F^+(t,\varphi)-F^+(s,\psi), \varphi(0)-\psi(0) >_+\leq M\|\varphi-\psi\| + L\big(\|\psi\|\big)\|g(t)-g(s)\| \quad (3.58)$$

and so condition (B4)(ii) is satisfied with $M=a_1+a_1\beta$, and $\|g(t)-g(s)\| = |a(t)-a(s)|\ +|\eta_t-\eta_s|\ +|a(t)b(t)-a(s)b(s)|$.
Using that $B(t)$ are m–completely accretive, and $0\in B(t)0$ we see that $J_\lambda^{B(t)}\hat{X}(t)\subset\hat{X}(t)$. Therefore by Lemma 3.14, to show the subtangential condition (3.16) for (FDE)$^+$, it is enough to prove that: for all $\psi\in\hat{E}(t)$, $t\geq 0$

$$\liminf_{\lambda\to 0^+}\tfrac{1}{\lambda}\,d(\psi(0)+\lambda F^+(t,\psi),[0,\beta(t+\lambda)])=0.$$

Starting from (3.54), it is not difficult to see that

$$\liminf_{\lambda\to 0^+}\frac{1}{\lambda}\,d(\psi(0)+\lambda F^+(t,\psi),[0,\beta(t+\lambda)])=\int_D F^+(t,\psi)-\beta'(t),$$

where $D=(\psi(0)=\beta(t))\cap(F^+(t,\psi)>\beta'(t))$.

According to (3.57), $F^+(t,\psi)\leq\beta'(t)$ on $(\psi(0)=\beta(t))$. So the above integral is zero, as desired. Therefore according to Theorem 3.9, given any $\varphi\in\hat{E}(t)$, there exists a global mild solution u_φ to (FDE)$^+$ with $(u_\varphi)_t\in\hat{E}(t)$ for all $t\geq 0$. But on $\hat{E}(t)$, $F^+(t,.)=F(t,.)$, $t\geq 0$. Thus u_φ is actually a mild solution to (3.56). From (3.58) and Theorem 3.9(ii), we also conclude that u_φ is unique amongst all solutions u to (3.56) with $u_t\geq 0$, and $0\leq u(t)\leq\beta(t)$, $t\geq 0$.

To prove assertion (ii)(a), we shall apply Corollary 4.17 with $g(t) = F(t,(u_\varphi)_t)$ to obtain

$$\left\| J_\lambda^{B(r_0)} u_\varphi(t) - u_\varphi(t) \right\| \leq \|u_\varphi(t+\lambda) - u_\varphi(t)\| + \frac{2}{\lambda}\int_t^{t+\lambda} \|u_\varphi(\tau) - u_\varphi(t)\| d\tau$$

$$+C_t \int_t^{t+\lambda} \|f(\tau) - f(r_0)\| d\tau + \int_t^{t+\lambda} \|F(\tau,(u_\varphi)_\tau)\| d\tau, \tag{3.59}$$

for all $t \geq 0$, $\lambda > 0$ and with $C_t = \max\{L\big(\left\| J_\lambda^{B(r_0)} u_\varphi(t)\right\|\big), L\big(\sup_{0\leq\tau}\|u_\varphi(\tau)\|\big)\}$. We note that $u_\varphi(\mathbb{R}^+)$ is L^∞–bounded, and so according to our assumption $J_\lambda^{B(r_0)}(u_\varphi(\mathbb{R}^+))$ is a relatively compact set in L^1. Moreover $\|F(t,(u_\varphi)_t)\|_\infty \leq a_1\beta$ for all $t \geq R$. Thus from uniform continuity of u_φ, and boundedness of f it follows that, for any $\varepsilon > 0$, the set $u_\varphi((R,\infty))$ is ε–close in L^1–norm to a relatively compact set. This implies that $u_\varphi(\mathbb{R}^+)$ is relatively compact in L^1.
Concerning (ii)(b), we first note that since the family $B(t)$ is m–accretive, it generates an evolution operator $U_B(t,s)$. We show that u_φ is uniformly continuous on $[T,\infty)$. Let $s,t \geq T$ such that $0 \leq t-s \leq T$. Then according to Lemma 4.21 (see Chapter 4);

$$\|u_\varphi(t) - u_\varphi(s)\| \leq \int_{kT+r_2}^{kT+r_1} \|F(\tau,(u_\varphi)_\tau)\| d\tau + \|U_B(r_1,r_2)u_\varphi(kT+r_2) - u_\varphi(kT+r_2)\|,$$

where $T \leq r_2 \leq r_1 \leq 3T$, and $k \in \mathbb{N}_0$.
Note once again, that both $u_\varphi(\mathbb{R}^+)$ and $\{F(t,(u\varphi)_t) \mid t \geq R\}$ are $L^\infty-$ bounded. So for any $r_2 \in [0,3T]$, $U_B(.,r_2)u_\varphi(\mathbb{R}^+)$ is uniformly equicontinuous on $[r_2,3T]$, and consequently, uniformly for $r_2 \in [T,3T]$. This implies that the last term in the above inequality tends to 0, as $t-s = r_1 - r_2 \to 0$. From this and the boundedness of F , we conclude that u_φ is uniformly continuous. Applying the result of (a) completes the proof of $(iii)(b)$.

Turning to the proof of (iii), let B be also α–accretive for some $\alpha \leq 0$. Since $[0,0] \in B$, from (1.9) we read that

$$e^{-\alpha t}\|u_\varphi(t)\| - \|u_\varphi(0)\| \;\; \leq \;\; \int_0^t e^{-\alpha\tau} < F(\tau,(u_\varphi)_\tau), u_\varphi(\tau) >_+ d\tau \tag{3.60}$$

for all $t \geq 0$. (Note that in (B1), $f = 0$). Set $N(t,x) = \|x\|$. Then applying (3.55) for $\beta = 0$, we have

$$
\begin{aligned}
&< F(\tau,(u_\varphi)_\tau), u_\varphi(\tau) >_+ = \int_{(u_\varphi(\tau)>0)} F(\tau,(u_\varphi)_\tau)d\tau \\
&= a(\tau)\int_{(u_\varphi(\tau)>0)} u_\varphi(\tau)[1 - b(\tau)u_\varphi(\tau) - G(\tau,(u_\varphi)_\tau)]d\tau \leq a(\tau)\|u_\varphi(\tau)\|,
\end{aligned}
$$

we conclude from (3.60) that

$$e^{-\alpha t}\|u_\varphi(t)\| \leq \|u_\varphi(0)\| + \int_0^t a(\tau)e^{-\alpha\tau}\|u_\varphi(\tau)\|d\tau \quad \text{for all} \quad t \geq 0.$$

An application of Gronwall's lemma completes the proof.

Proof of proposition 3.29. According to Proposition 3.26 (i), u_φ exists and $u_\varphi(t) \in [0, \max\{\beta_0, \|\varphi(0)\|_\infty]$ for all $t \geq 0$. To prove the first assertion, we shall first show that $d\big(u_\varphi(t),[0,\beta_0]\big)$ is non–increasing. To this aim, let $N(t,x) = d\big(x,[0,\beta_0]\big)$, and $M(t,x) = 0$. Then $\limsup_{h\to 0^+} \frac{N(t-h,x)-N(t,x)}{h} = 0$, and therefore by Lemma 3.21, M and N satisfy (3.47)(we note that $\alpha = 0$). Thus, Theorem 3.22 implies that

$$d\big(u_\varphi(t),[0,\beta_0]\big) \leq d\big(u_\varphi(s),[0,\beta_0]\big) + \int_s^t \partial N(\tau, u_\varphi(\tau), F(\tau,(u_\varphi)_\tau)d\tau. \tag{3.61}$$

From (3.55),

$$\partial N(\tau, u_\varphi(\tau), F(\tau, u_\varphi(\tau)) = \int_{Q_\tau} F(\tau,(u_\varphi)_\tau) \leq 0, \tag{3.62}$$

where $Q_\tau = (u_\varphi(\tau) > \beta_0)$. Therefore by (3.61), $d\big(u_\varphi(t),[0,\beta_0]\big) \downarrow \delta \geq 0$.
By (3.54), $d\big(u_\varphi(\tau),[0,\beta_0]\big) = \int_{Q_\tau}(u_\varphi(\tau) - \beta_0)$ for all $\tau \geq 0$. Then

$$
\begin{aligned}
0 \quad &\leq \quad a_0(t-s)\delta \leq \int_s^t a(\tau)d\left(u_\varphi(\tau),[0,\beta_0]\right)d\tau = \int_s^t a(\tau)\Big(\int_{Q_\tau}(u_\varphi(\tau) - \beta_0)\Big)d\tau \\
&\leq \quad \int_s^t a(\tau)\Big(\int_{Q_\tau}(u_\varphi(\tau) - \frac{1}{b(\tau)}\Big)d\tau = \int_s^t a(\tau)\Big(\int_{Q_\tau}\frac{1}{b(\tau)}\big[b(\tau)u_\varphi(\tau) - 1\big]\Big)d\tau \\
&\leq \quad \int_s^t \Big(\int_{Q_\tau} a(\tau)u_\varphi(\tau)\big[b(\tau)u_\varphi(\tau) - 1 + G(\tau,(u_\varphi)_\tau)\big]\Big)d\tau.
\end{aligned}
$$

The above inequality together with (3.62), and (3.61) imply that

$$0 \leq a_0(t-s)\delta \leq d\left(u_\varphi(s),[0,\beta_0]\right) - \delta \quad \text{for all} \quad 0 \leq s \leq t.$$

This shows that $\delta = 0$, and so (i) holds. Assertion (ii) is now obvious. Since for a finite measure space, L^∞–order intervals are L^1 weakly relatively compact, the assertion (iii) is then a consequence of part (i).

Consider the following variant of model (3.56), described in Section 2.6.

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni u(t)\left[1 + a(t)u(t) - b(t)(u(t))^2 \right. \\ \qquad\qquad \left. -(1 + a(t) - b(t)) \int_{-r}^{0} f(s)u(t+s)ds\right], \quad t \geq 0 \\ u_{|[-r,0]} = \varphi \end{cases} \tag{3.63}$$

Proposition 3.30. *Let $B(t) \subset L^1(\Omega) \times L^1(\Omega)$ be a family of m-completely accretive operators satisfying condition (B1) with $0 \in B(t)0$ for all $t \geq 0$. Consider the above delay equation with initial–history space $E := C([-R, 0]; X)$. Assume $\beta : \mathbb{R}^+ \to (0, \infty)$ is a bounded differentiable function such that β' is also bounded and it satisfies the following property:*

$$\beta(t)[1 + a(t)\beta(t) - b(t)\beta^2(t)] \leq \beta'(t) \quad \textit{for all} \quad t \geq 0. \tag{3.64}$$

Then for $\varphi \in C([-R, 0]; X)$ with $\varphi(s) \geq 0$, $s \in [-R, 0]$, $\varphi(0) \in [0, \beta(0)]$ a.e. on Ω, and $\varphi(0) \in cl(D(B(0)))$ all the assertions of Proposition 3.26 *hold, with $a(\tau)$ in (iii) being replaced by $1 + a(\tau)\beta(\tau)$.*

Remark 3.31. Under the assumptions of the above proposition, let β be any bounded nondecreasing differentiable function such that $\beta(t) \geq \frac{a(t)+\sqrt{a(t)^2+4b(t)}}{2b(t)}$, and β' is bounded. Then the conclusions of Proposition 3.26 hold.

4 Asymptotic behavior of solutions to (FDE)

In this chapter we supplement the general existence and flow invariance results of the foregoing chapters by some specific results on the asymptotic behavior of solutions, such as asymptotic stability, almost periodicity and compactness.

4.1 Asymptotic stability of solutions to (FDE)

Let u_φ and u_ψ be mild solutions to (FDE) under the assumptions of Theorem 2.2 or Theorem 3.9. Then $\|(u_\varphi)_t - (u_\psi)_t\| \leq e^{\omega(t-s)}\|\varphi - \psi\|$, $\omega = \max\{0, \alpha + M\}$. This result can be improved in particular cases:
(a) In the infinite delay case
1. For $E = (BUC(\mathbb{R}^-; X), sup - norm)$, $\alpha + M < 0$ does not imply asymptotic stability for (FDE) [61, Example 4.1.A].
2. For the E_v spaces where $s \mapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$ for some $\mu > 0$, and $\alpha + M < 0$, solutions to the autonomous (FDE) are exponentialy asymptotically stable, [61, Theorem]. For the case $v(s) = e^s$, see also [50].
We will show that in the space E_v as in 2, the solutions to the non-autonomous (FDE) are also exponentially asymptotically stable.

Theorem 4.1. *Assume that $v : \mathbb{R}^- \to (0, 1]$ is a weight satisfying $(v1)$ and $(v2)$ such that $s \mapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$ for some $\mu > 0$, and put $\beta = \min\{\mu, -\alpha - M\}$. Then the evolution operator $(U(t, s))_{t \geq s}$ for (FDE) via Proposition 2.3 with initial space E_v, satisfies*

$$\|U(t, s)\varphi - U(t, s)\psi\|_v \leq e^{-\beta(t-s)}\|\varphi - \psi\|_v$$

for all $t \geq s$, and $\varphi, \psi \in cl(D(A(s)))$. In particular,

$$\|u_\varphi\varphi(t) - u_\varphi\psi(t)\| \leq e^{-\beta(t-s)}\|\varphi - \psi\|_v,$$

for all $t \geq s$, and $\varphi, \psi \in cl(D(A(s)))$.

Proof. According to (1.7) it will suffice to show that the family $A(t)$ defined by (2.3) satisfies

$$(1+\lambda\beta)\|\varphi_1-\varphi_2\|_v \le \|(\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')\|_v + \lambda H(t_1,t_2)L(\|\varphi_2\|_v) \tag{4.1}$$

for all $\lambda > 0$ with $-\lambda\beta < 1$, $\varphi_i \in D(A(t_i))$, $i \in \{1,2\}$, $t_2 \le t_1$, and H as in (2.3). Let $x \in X$, $\psi \in E_v$, and $\lambda > 0$. As in the proof of [61, Theorem 3.6], using that the function $s \mapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$ we have

$$v(\theta)\|\varphi(\theta)\| \;\le\; \frac{1}{1+\lambda\mu}\|\psi\|_v + e^{\frac{(1+\lambda\mu)}{\lambda}\theta}\left(\|x\| - \frac{1}{1+\lambda\mu}\|\psi\|_v\right). \tag{4.2}$$

Now take $\lambda > 0$ such that $-\lambda\beta < 1$, and let $\varphi_i \in D(A(t_i))$, and put $\psi_i = (I + \lambda A(t_i))\varphi_i$, $i \in \{1,2\}$. Then

$$(\varphi_1-\varphi_2)-\lambda(\varphi_1'-\varphi_2') = (\psi_1-\psi_2), \tag{4.3}$$

and therefore (4.2) implies that for all $\theta \le 0$

$$\begin{aligned} v(\theta)\|(\varphi_1-\varphi_2)(\theta)\| \;&\le\; \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_v \\ &+\; e^{\frac{(1+\lambda\mu)}{\lambda}\theta}\left(\|\varphi_1(0)-\varphi_2(0)\| - \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_v\right). \end{aligned} \tag{4.4}$$

If $\|\varphi_1(0)-\varphi_2(0)\| \le \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_v$, then we read from (4.4) that

$$\|\varphi_1-\varphi_2\|_v \le \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_v,$$

and therefore

$$(1+\lambda\beta)\|\varphi_1-\varphi_2\|_v \le \|\psi_1-\psi_2\|_v.$$

Hence (4.1) is obviously true. On the other hand, if $\|\varphi_1(0)-\varphi_2(0)\| > \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_v$, then we conclude from (4.4) that

$$\|\varphi_1-\varphi_2\|_v \le \|\varphi_1(0)-\varphi_2(0)\|. \tag{4.5}$$

Note that $[\varphi_i(0), F(t,\varphi_i)-\varphi_i'(0)] \in B(t_i), i \in \{1,2\}$ and $\lambda\alpha < 1$. Thus using (2.1) and following the method of proof in Proposition 2.3 we obtain

$$\begin{aligned} (1-\lambda(\alpha+M))\|\varphi_1-\varphi_2\|_v \;&\le\; \|\psi_1-\psi_2\|_v \\ &+\; \lambda L(\|\varphi_2\|_v)\big(\|f(t_1)-f(t_2)\| + \|g(t_1)-g(t_2)\|\big), \end{aligned}$$

therefore by our choice for β, inequality (4.1) is satisfied.

Corollary 4.2. *Under the assumption of Theorem 4.1, if $M+\alpha<0$, then solutions to (FDE) are exponentially asymptotically stable.*

(b) In the finite delay case for $E=C([-R,0];X)$, Plant [50] shows that if $\alpha+M<0$ then the solutions to the autonomous (FDE) are exponentially asymptotically stable. In [13], it has been proved that, if X^* is a uniformly convex Banach space and $\alpha+M<0$, then classical solutions to the non-autonomous (FDE) are asymptotically stable.
Let $E=C([-R,0];X)$ with $\|.\|_\infty$. For $\varphi\in E$ and $\mu>0$ define

$$\|\varphi\|_\mu=\sup_{\theta\in[-R,0]} e^{\mu\theta}\|\varphi(\theta)\|.$$

It can easily be seen that

$$\|\varphi\|_\infty\le e^{\mu R}\|\varphi\|_\mu\le e^{\mu R}\|\varphi\|_\infty. \tag{4.6}$$

$$\text{Assume that condition (B.2)}(ii)\quad\text{holds for all}\quad \varphi\in\hat{E}(t_1)\text{ and }\psi\in\hat{E}(t_2). \tag{4.7}$$

We then apply the techniques of proof in [50] to show

Proposition 4.3. *Let $E=\big(C([-R,0];X),\|.\|_\infty\big)$. If $\alpha+M<0$, then there exist K and $\beta>0$ such that for all $\varphi,\psi\in cl(D(A(s)))$,*

$$\|U(t,s)\varphi-U(t,s)\psi\|_\infty\le Ke^{-\beta(t-s)}\|\varphi-\psi\|_\infty.$$

Proof. Choose $\mu>0$ such that $\alpha+e^{\mu R}M<0$. Set $\beta=\min\{\mu,-(\alpha+e^{\mu R}M)\}$. We will first show that for $\lambda>0$ and for all $s\le t_2\le t_1$, and $\varphi_i\in D(A(t_i)), i\in\{1,2\}$,

$$(1+\lambda\beta)\|\varphi_1-\varphi_2\|_\mu\le\|(\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')\|_\mu+\lambda\|f(t_1)-f(t_2)\|L(\|\varphi_2\|_\mu). \tag{4.8}$$

Take $\lambda>0$ and $\varphi_i\in D(A(t_i))$, $i\in\{1,2\}$. Put $\psi_i=(I+\lambda A(t_i))\varphi_i$. Then applying (4.2) for $v(\theta)=e^{\mu\theta}$ and $\theta\in[-R,0]$ we have

$$\begin{aligned} e^{\mu\theta}\|(\varphi_1-\varphi_2)(\theta)\| &\le \frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_\mu \\ &+ e^{\frac{(1+\lambda\mu)}{\lambda}\theta}\left(\|\varphi_1(0)-\varphi_2(0)\|-\frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_\mu\right). \end{aligned} \tag{4.9}$$

If $\|\varphi_1(0)-\varphi_2(0)\|\le\frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_\mu$, then $\|\varphi_1-\varphi_2\|_\mu\le\frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_\mu$. Since $\beta\le\mu$, the inequality (4.8) is obviously satisfied.
In the case $\|\varphi_1(0)-\varphi_2(0)\|>\frac{1}{1+\lambda\mu}\|\psi_1-\psi_2\|_\mu$, (4.9) implies that

$$\|\varphi_1-\varphi_2\|_\mu\le\|\varphi_1(0)-\varphi_2(0)\|.$$

Therefore using (2.1) for $[\varphi_i(0), F(t_i,\varphi_i) - \varphi_i'(0)]$, $i \in \{1,2\}$ we obtain

$$\begin{aligned}(1-\lambda\alpha)\|\varphi_1-\varphi_2\|_\mu &\le \|(\varphi_1(0)-\lambda\varphi_1'(0))-(\varphi_2(0)-\lambda\varphi_2'(0))\|\\ &+ \lambda\|(F(t_1,\varphi_1)-F(t_2,\varphi_2)\| + \lambda L_1(\|\varphi_2(0)\|)\|f(t_1)-f(t_2)\|.\end{aligned}$$

Now (4.7) implies

$$\begin{aligned}(1-\lambda\alpha)\|\varphi_1-\varphi_2\|_\mu &\le \|(\varphi_1-\lambda\varphi_1')-(\varphi_2-\lambda\varphi_2')\|_\mu + \lambda\|\varphi_1-\varphi_2\|_\infty\\ &+ \lambda\|g(t_1)-g(t_2)\|L_2(\|\varphi_2\|_\infty) + \lambda L_1(\|\varphi_2\|_\mu)\|f(t_1)-f(t_2)\|,\end{aligned}$$

and therefore from (4.6) we obtain

$$\begin{aligned}(1+\lambda(-\alpha-e^{\mu R}M))\|\varphi_1-\varphi_2\|_\mu &\le \|\psi_1-\psi_2\|_\mu + \lambda\|g(t_1)-g(t_2)\|L_2(e^{\mu R}\|\varphi_2\|_\mu)\\ &+ \lambda L_1(\|\varphi_2\|_\mu)\|f(t_1)-f(t_2)\|.\end{aligned}$$

We now define $L : \mathbb{R}^+ \to \mathbb{R}^+$ by $L(x) = L_1(x) + L_2(e^{\mu R}x)$, and H as in (2.3). Since $\beta \le -\alpha - e^{\mu R}M$, we obtain the desired inequality. By Proposition 2.3

$$\|U(t,s)\varphi - U(t,s)\psi\|_\mu \le e^{-\beta(t-s)}\|\varphi-\psi\|_\mu.$$

Let $K = e^{\mu R}$ then (4.6) completes the proof.

Remark 4.4. The results of this section hold as well for $U(t,s)$ the evolution operator associated to $A(t)$ defined by (3.24) and u_φ, the mild solution to (FDE), as in Theorem 3.9, with α in (B1) and M in (B4)(ii). Actually the proofs are based on the argument used in (S1)1. in the proof of Theorem 3.9.

4.2 Compact evolution operators

In this section we shall consider (FDE) in the initial history space $E = C([-R,0];X)$. We study the relation between $U_B(t,s)$ the evolution operator generated by $B(t)$, and $U_A(t,s)$ the evolution operator generated by $A(t)$, for $A(t)$ as in (2.3), or (3.24). More precisely we show that if $U_B(t,s)$ is a compact evolution operator, then $U_A(t,s)$ is also compact for all $t > R+s$.

We shall use the following to determine (weakly) relatively compact sets.

Lemma 4.5. *Let X be a Banach space. If $B \subset X$ is such that for every $\varepsilon > 0$, there exists a (weakly) relatively compact subset $B_\varepsilon$in X such that $B \subset B_\varepsilon + \varepsilon B_X$, then B is (weakly) relatively compact.*

Proof. Showing that B is relatively compact is trivial. For the weakly relatively compact part see [24, P 221, Lemma 2].

Theorem 4.6. *Let $(B(t))_{t\geq 0}$ be a family of operators satisfying* (2.1)*, and such that $(B(t))_{t\geq 0}$ generates an evolution operator $U_B(t,s) : cl(D(B(s))) \to cl(D(B(t)))$, $t \geq s$. Assume, moreover, that $U_B(t,s)$ is a compact evolution operator. Then for any $t > R+s$, the evolution operator $U_A(t,s)$ generated by $A(t)$ as in Theorem 3.9, or Theorem 2.2 is a compact operator.*

Proof. Let $t > s+R$, and let K be a bounded subset of $cl(D(A(s)))$. We need to show that $U_A(t,s)K$ is a relatively compact subset of $C([-R,0];X)$. To this aim, we use the Arzelá–Ascoli Theorem, and equivalently prove:
(i) For all $t_0 \in [-R,0]$, the cross section of $U_A(t,s)K$ at t_0, i.e $\{(U_A(t,s)\varphi)(t_0) \mid \varphi \in K\}$ is relatively compact in X.
(ii) $U_A(t,s)K$ is equicontinuous at t_0 for all $t_0 \in [-R,0]$.

To prove (i), let $t_0 \in [-R,0]$. Then $\{U_A(\tau,s)\varphi) \mid \varphi \in K,\ \tau \in [s,t+t_0]\}$ is bounded. Indeed, fix $\varphi_0 \in K$. Then

$$\begin{aligned} \|U_A(\tau,s)\varphi)\| &\leq \|U_A(\tau,s)\varphi) - U_A(\tau,s)\varphi_0)\| + \|U_A(\tau,s)\varphi_0)\| \\ &\leq e^{\omega(\tau-s)}\|\varphi-\varphi_0\| + \|U_A(\tau,s)\varphi_0)\|, \end{aligned}$$

for all $\varphi \in K$, and $\tau \in [s,t+t_0]$. F maps bounded sets to bounded sets, thus we may set

$$M = \sup\{\|F(\tau, U_A(\tau,s)\varphi)\| \mid \varphi \in K, \tau \in [s,t+t_0]\}.$$

Since $t > R+s$, we may choose $\delta_\varepsilon < \varepsilon(e^{|\alpha|\ (t+t_0)}M)^{-1}$ such that $t+t_0-\delta_\varepsilon > s$. Then

$$\begin{aligned} &\left\|(U_A(t,s)\varphi)(t_0) - U_B(t+t_0,t+t_0-\delta_\varepsilon)u_\varphi(t+t_0-\delta_\varepsilon)\right\| = \\ &\quad \left\|u_\varphi(t+t_0) - U_B(t+t_0,t+t_0-\delta_\varepsilon)u_\varphi(t+t_0-\delta_\varepsilon)\right\| \leq \\ &\qquad \int_{t+t_0-\delta_\varepsilon}^{t+t_0} e^{\alpha(t+t_0-\tau)}\|F(\tau, U_A(t+\tau,s)\varphi\|d\tau. \end{aligned}$$

Since $\{u_\varphi(t+t_0-\delta_\varepsilon) \mid \varphi \in K\}$ is bounded, and $U_B(t+t_0,t+t_0-\delta_\varepsilon)$ is a compact operator, from the above inequality it follows that, given any $\varepsilon > 0$, $\{(U_A(t,s)\varphi)(t_0) \mid \varphi \in K\}$ is ε–close to a relatively compact set and thus is relatively compact.

Returning to (ii), take $t_0 \in [-R,0]$, and let $\varepsilon > 0$. We shall prove that $U(t,s)K$ is equicontinuous at t_0. Since $t > R+s$, we can find $0 < \delta < 1$ such that $t-\delta > R+s$, and $t + t_0 - \delta > s$. Then we have

$$\begin{aligned}\|(U_A(t,s)\varphi)(t_0+h) - (U_A(t,s)\varphi)(t_0)\| = \big\|u_\varphi(t+t_0+h) - u_\varphi(t+t_0)\big\| \le \\ \big\|u_\varphi(t+t_0+h) - U_B(t+t_0+h, t+t_0-\delta)u_\varphi(t+t_0-\delta)\big\| + \\ \big\|U_B(t+t_0, t+t_0-\delta)u_\varphi(t+t_0-\delta) - u_\varphi(t+t_0)\big\| + \\ \big\|U_B(t+t_0+h, t+t_0-\delta)u_\varphi(t+t_0-\delta) - U_B(t+t_0, t+t_0-\delta)u_\varphi(t+t_0-\delta)\big\|\end{aligned}$$

for all $\varphi \in K$, and $|h| < \delta$. Therefore

$$\begin{aligned}&\|(U_A(t,s)\varphi)(t_0+h) - (U_A(t,s)\varphi)(t_0)\| \le \\ &\int_{t+t_0-\delta}^{t+t_0+h} e^{\alpha(t+t_0+h-\tau)}\|F(\tau, U_A(\tau,s)\varphi\|d\tau + \int_{t+t_0-\delta}^{t+t_0} e^{\alpha(t+t_0-\tau)}\|F(\tau, U_A(\tau,s)\varphi\|d\tau + \\ &\big\|U_B(t+t_0+h, t+t_0-\delta)u_\varphi(t+t_0-\delta) - U_B(t+t_0, t+t_0-\delta)u_\varphi(t+t_0-\delta)\big\|\end{aligned} \tag{4.10}$$

Set $M = \sup\{F(\tau, U_A(\tau,s))\varphi \,|\, \varphi \in K, \tau \in [t+t_0-1, t+t_0+1]\}$. We note that $u_\varphi(t+t_0-\delta) = (u_\varphi)_{(t-\delta)}(t_0) = (U_A(t-\delta,s)\varphi)(t_0)$, and so by the first assertion the set $\{u_\varphi(t+t_0-\delta) \,|\, \varphi \in K\}$ is a relatively compact subset of $cl(D(B(t+t_0-\delta)))$. Thus there exists $\delta_\varepsilon < \min\{\delta, \varepsilon(3e^{|\alpha|\ (t+t_0+1)}M)^{-1}\}$, such that for all h, $|h| < \delta_\varepsilon$, and $\varphi \in K$

$$\big\|U_B(t+t_0+h, t+t_0-\delta)u_\varphi(t+t_0-\delta) - U_B(t+t_0, t+t_0-\delta)u_\varphi(t+t_0-\delta)\big\| < \varepsilon/3.$$

Therefore

$$\|(U_A(t,s)\varphi)(t_0+h) - (U_A(t,s)\varphi)(t_0)\| < \varepsilon,$$

for all h with $|h| < \delta_\varepsilon$, and $\varphi \in K$ as desired.

Remark 4.7. If, in Proposition 4.6, we assume that $U_B(t,s)$ is equicontinuous, then for any $t > R+s$, and any bounded subset K of $cl(D(A(s)))$, the family $\{U_A(t,s)\varphi \,|\, \varphi \in K\}$ is equicontinuous on $[-R, 0]$. To see this, we shall start with the same argument as the second part of the above proof. Then $\{u_\varphi(t+t_0-\delta) \,|\, \varphi \in K\}$ is bounded. Thus by equicontinuity of $U_B(., t+t_0-\delta)$ at $t+t_0$, we may choose δ_ε sufficiently small such that the righthand side of (4.10) is less than ε.

4.3 Almost periodicity properties of solutions to (FDE)

In this section we will study the relationship between properties of u_φ, the mild solution to (FDE), and the corresponding motion $U(.,0)\varphi : \mathbb{R}^+ \to E$. In particular we discuss almost periodicity properties of solutions to (FDE). Our study is based on the representation $u_\varphi(t) = (U(t,0)\varphi)(0)$ for $\varphi \in cl(D(A(0)))$, and $t \geq 0$. For the autonomous (FDE), see [61].

We start by recalling the required periodicity concepts. For $J \in \{\mathbb{R}, \mathbb{R}^+\}$, we let $C_b(J,X)$ denote the Banach space of bounded continuous functions from J into X with the supremum norm $\|.\|_\infty$, while $BUC(\mathbb{R}^+,X)$, respectively $C_0(\mathbb{R}^+,X)$, denote the subspace consisting of those $f \in C_b(\mathbb{R}^+,X)$ which are uniformly continuous, respectively vanish at ∞. Further, given a function $f : J \to X$ and $\omega \in J$, the $\omega-$translate f_ω of f is defined by $f_\omega(t) = f(t+\omega), t \in J$, and $H(f) = \{f_\omega \,|\, \omega \in J\}$ will denote the set of all translates of f.

Definition 4.8. (i) A function $f \in C_b(\mathbb{R},X)$, (respectively $f \in C_b(\mathbb{R}^+,X)$) is said to be *almost periodic* ,(respectively *asymptotically almost periodic*) if $H(f)$ is a relatively compact set in $C_b(\mathbb{R},X)$(respectively $C_b(\mathbb{R}^+,X)$) with respect to the supremum norm.
(ii) A function $f \in C_b(\mathbb{R}^+,X)$ is said to be *Eberlein – weakly almost periodic* if $H(f)$ is weakly relatively compact in $C_b(\mathbb{R}^+,X)$.

The spaces of $X-$valued functions defined in (i) and (ii) of Definition 4.8 will be denoted respectively by (i) $AP(\mathbb{R},X)$, $(AAP(\mathbb{R}^+,X))$, and (ii) $W(\mathbb{R}^+,X)$. We also let $W_0(\mathbb{R}^+,X)$ denote the vector space of $W(\mathbb{R}^+,X)$ consisting of those $\varphi \in W(\mathbb{R}^+,X)$ for which the zero function belongs to the weak closure of $H(\varphi)$.
For later use, we also need the following basic facts about the above listed concepts of almost periodicity (see Fréchet [21, 22], DeLeeuw-Glicksburg [10, 11] and [62, 63, 64, 65]):
1. $f \in C_b(\mathbb{R}^+,X)$ is asymptotically almost periodic (respectively, Eberlein-weakly almost periodic) if and only if there exist unique functions $g \in AP(\mathbb{R},X)$ and $\varphi \in C_0(\mathbb{R}^+,X)$ (respectively, $\varphi \in W_0(\mathbb{R}^+,X)$) such that $f = g|_{\mathbb{R}^+} + \varphi$.

2. A function f is is asymptotically almost periodic if for every $\varepsilon > 0$, there exist $M_\varepsilon > 0$, and a relatively dense subset $P_\varepsilon \subset \mathbb{R}^+$ such that

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all} \quad t \geq M_\varepsilon \quad \text{and} \quad \tau \in P_\varepsilon.$$

Recall that a subset P of $\mathbb{R}^+$ is called relatively dense if there exists $l = l(P) > 0$ such that $[a, a+l] \cap P \neq \emptyset$ for all $a \in \mathbb{R}^+$. Finally we shall say that a function $f : \mathbb{R}^+ \to X$ is almost periodic if it is the restriction to $\mathbb{R}^+$ of an almost periodic function $g : \mathbb{R} \to X$.

Let $U(t,0)$ be the evolution operator generated by $A(t)$, and u_φ be a mild solution of (FDE). As

$$U(t,0)\varphi = (u_\varphi)_t \quad \text{for all} \quad t \geq 0, \tag{4.11}$$

asymptotic properties of the motion $U(.,0)\varphi : \mathbb{R}^+ \to E$ (such as having relatively (weakly relatively) compact range, asymptotic almost periodicity, and Eberlein-weak almost periodicity) can directly carry over to corresponding asymptotic properties of u_φ. In this section we are interested to find out under which conditions, if these asymptotic properties hold for u_φ then they hold as well for the motion $U(.,0)\varphi$. We shall look at both the finite delay and the infinite delay case:

1. Consider (FDE) in the finite delay case with $E = C([-R,0];X)$. Then

Theorem 4.9. *Given $\varphi \in cl(D(A(0)))$, the following assertions hold:*
(i) If u_φ is uniformly continuous and has relatively compact range in X, then $\{U(t,0)\varphi \,|\, t \geq 0\}$ is relatively compact in $C([-R,0];X)$.
(ii) If $u_\varphi|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$, then $U(.,0)\varphi \in AAP(\mathbb{R}^+, C([-R,0];X))$.

Proof. To prove (i), let u_φ be uniformly continuous, and the set $\{u_\varphi(t) \,|\, t \in \mathbb{R}^+\}$ be relatively compact. Take $t_0 \in [-R,0]$, Then

$$\{(U(t,0)\varphi)(t_0) \,|\, t \in \mathbb{R}^+\} = \{u_\varphi(t) \,|\, t \in \mathbb{R}^+\} \cup \varphi([t_0,0]), \tag{4.12}$$

and so the set $\{(U(t,0)\varphi)(t_0) \,|\, t \in \mathbb{R}^+\}$ is relatively compact.
Choose $\delta > 0$ sufficiently small. Then for all h, $|h| < \delta$ we have

$$\|(U(t,0)\varphi)(t_0+h) - (U(t,0)\varphi)(t_0)\| = \begin{cases} \|u_\varphi(t+t_0+h) - u_\varphi(t+t_0)\|, & t+t_0 > 0 \\ \|u_\varphi(h) - \varphi(0)\|, & t = -t_0 \\ \|\varphi(t+t_0+h) - \varphi(t+t_0)\|, & -R \leq t+t_0 < 0. \end{cases}$$

Since u_φ is uniformly continuous on $\mathbb{R}^+$, and φ is continuous on $[-R,0]$, it follows that the family $\{U(t,s)\varphi \,|\, t \in \mathbb{R}^+\}$ is equicontinuous at any $t_0 \in [-R,0]$. Now applying Arzelá-Ascoli's Theorem we conclude that $U(\mathbb{R}^+,0)\varphi$ is relatively compact in $C([-R,0])$.

For the second assertion, assume $u_\varphi|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$. Given $\varepsilon > 0$ there exist a relatively dense subset $P_\varepsilon \subset \mathbb{R}^+$, and $T_\varepsilon > 0$ such that

$$\|u_\varphi(t+\tau) - u_\varphi(t)\| < \varepsilon/2 \quad \text{for all} \quad \tau \in P_\varepsilon \quad \text{and} \quad t \geq T_\varepsilon.$$

Thus for all $t \geq T_\varepsilon + R$ and $\tau \in P_\varepsilon$

$$\|U(t+\tau,0)\varphi - U(t,0)\varphi\|_\infty = \sup_{-R\leq s\leq 0} \|u_\varphi(t+\tau+s) - u_\varphi(t+s)\| < \varepsilon,$$

so $U(.,0)\varphi$ is also asymptotically almost periodic.

In order to prove weak compactness we shall use of the following criterion:

Proposition 4.10. (Ruess/Summers) *Let (T,τ) be a completely regular topological space. A subset $H \subset C_b(T,X)$, is weakly relatively compact if and only if,*
(i) H is bounded in $C_b(T,X)$, and
(ii) for all $\{h_m\}_{m\in\mathbb{N}} \subset H$, $\{t_n\}_{n\in\mathbb{N}} \subset T$, and $\{x_n^*\}_{n\in\mathbb{N}} \subset extB_{X^*}$ the following double limit condition holds:

$$\lim_m \lim_n < h_m(t_n), x_n^* >= \lim_n \lim_m < h_m(t_n), x_n^* >,$$

whenever the iterated limits exists.

Here, $extB_{X^*}$ denotes the set of extreme points of the dual unit ball of X.

Theorem 4.11. *Let $\varphi \in cl(D(A(0)))$, then the following assertions hold:*
(i) If u_φ is uniformly continuous and has weakly relatively compact range in X, then $\{U(t,0)\varphi \,|\, t \geq 0\}$ is weakly relatively compact in $C([-R,0];X)$.
(ii) If $u_\varphi|_{\mathbb{R}^+} \in W(\mathbb{R}^+,X)$, then $U(.,0)\varphi \in W(\mathbb{R}^+, C([-R,0];X))$.

Proof. To show the first assertion, let u_φ be a uniformly continuous solution to (FDE) with weakly relatively compact range in X. Take sequences (t_m) in $\mathbb{R}^+$ and $(s_n, x_n^*) \subset [-R,0] \times B_{X^*}$ such that $\alpha = \lim_n \lim_m < (U(t_m,0)\varphi)(s_n), x_n^* >$ and $\beta = \lim_m \lim_n < (U(t_m,0)\varphi)(s_n), x_n^* >$ both exist. We have to show that $\alpha = \beta$. Note that we only need to show the equality of the double limit, so we may pass to subsequences. Let (t_m) be bounded. We may assume that $t_m \to t_0$. Then $\|U(t_m,0)\varphi - U(t_0,0)\varphi\| \to 0$, and $\alpha = \lim_n < (U(t_0,0)\varphi)(s_n), x_n^* >$. Now, given $\varepsilon > 0$, choose $m_0 \in \mathbb{N}$ such that $\|(U(t_m,0)\varphi)(\tau) - (U(t_0,0)\varphi)(\tau)\| < \varepsilon$ for all $m \geq m_0$ and all $\tau \in [-R,0]$. Putting $\beta_m = \lim_n < (U(t_m,0)\varphi)(s_n), x_n^* >$, $m \in \mathbb{N}$, we have that

$$|\beta_m - \alpha| \; = \lim_n |< (U(t_m,0)\varphi)(s_n) - (U(t_0,0)\varphi)(s_m), x_n^* >| \; \leq \varepsilon$$

for all $m \geq m_0$. This implies that $\alpha = \beta$.
Assume (t_m) is unbounded. Let $s_n \to s$. We can assume $(t_m) \uparrow \infty$, as well as

$R \le t_1 \le t_m$. Since u_φ is uniformly continuous and $u_\varphi(\mathbb{R}^+)$ is weakly relatively compact in X, we can further assume that a subnet of $(u_\varphi|_{\mathbb{R}^+})_{t_m}$ converges pointwise weakly to some $h \in C(\mathbb{R}^+; X)$. Finally, some subnet of (x_n^*) clusters weak–star at some $x^* \in B_{X^*}$. Hence

$$\begin{aligned}\alpha &= \lim_n \lim_m < (U(t_m,0)\varphi)(s_n), x_n^* >= \lim_n \lim_m < u_{\varphi_{t_m}}(s_n), x_n^* > \\ &= \lim_n < h(s_n), x_n^* >=< h(s), x^* >= \lim_m < u_{\varphi_{t_m}}(s), x^* >= \beta.\end{aligned}$$

Applying Proposition 4.10, then completes the proof of (i).

To prove (ii), let $u_\varphi|_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$. Then according to [66, Proposition 2.1], $u_\varphi|_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$ is uniformly continuous. Using the relation (4.11), we can see that $U(.,0)\varphi : \mathbb{R}^+ \to C([-R,0]; X)$ is also uniformly continuous. We need to prove that $H = \{U_t(.,0)\varphi \,|\, t \ge 0\} \subset C_b(\mathbb{R}^+, C([-R,0]; X))$ is weakly relatively compact. Take the sequences $U_{\omega_n}(.,0)\varphi$ in H, (t_m) in $\mathbb{R}^+$, and $(s_m, x_m^*) \subset [-R,0] \times B_{X^*}$ such that the following double limits exist:

$$\alpha = \lim_n \lim_m < (U_{\omega_n}(t_m,0)\varphi)(s_m), x_m^* > \text{ and } \beta = \lim_m \lim_n < (U_{\omega_n}(t_m,0)\varphi)(s_m), x_m^* > . \tag{4.13}$$

By Proposition 4.10, we need to show that $\alpha = \beta$.

Case 1. If (ω_n) or (t_m) is bounded, then using the uniform continuity of $U(.,0)\varphi$ and following the argument in the first part, we can see that $\alpha = \beta$.

Case 2. Let (t_n) be unbounded. Since $(s_m) \subset [-R,0]$, we may choose a subsequence of t_m, denoted again by t_m such that $r_m = s_m + t_m \ge 0$. Then

$$\begin{aligned}&< (U_{\omega_n}(t_m,0)\varphi)(s_m), x_m^* >=< (U(\omega_n + t_m,0)\varphi)(s_m), x_m^* > \\ &\qquad =< u_\varphi(\omega_n + s_m + t_m, 0, x_m^* >=< (u_\varphi)_{\omega_n}(r_m), x_m^* >,\end{aligned}$$

therefore by (4.13), $\lim_n \lim_m < (u_\varphi)_{\omega_n}(r_m), x_m^* >$, and $\lim_m \lim_n < (u_\varphi)_{\omega_n}(r_m), x_m^* >$ exist. Now since $\{(u_\varphi)_t \,|\, t \ge 0\}$ is weakly relatively compact, using Proposition 4.10 we conclude that $\alpha = \beta$. (The case where (w_n) is unbounded is quite similar).

2. In this part we consider (FDE) in the context of the initial history spaces E_v with v satisfying (v1) and (v2). In addition to the assumptions (v1) and (v2), which will be assumed throughout this section, we shall need the following special properties of the weight function v:

$$(v3) \quad \lim_{s\to-\infty} v(s) = 0; \qquad\qquad (v3^*) \quad \lim_{t\to\infty} \sup_{s\le -t} \frac{v(s)}{v(s+t)} = 0.$$

Clearly, (v3*) implies (v3). If, for $\mu > 0$, $v_1(s) = e^{\mu s}$, and $v_2(s) = (1 + |s|\ )^{-\mu}$, $s \leq 0$, then v_1 and v_2 both fulfill conditions (v1), (v2), and (v3), v_1 fulfills (v3*), but v_2 fails to satisfy (v3*).

Following [61], we shall need the following lemma.

Lemma 4.12. *Given $\varphi \in E_v$ and $t \geq 0$, define $\tilde{\varphi} : \mathbb{R}^+ \to C(\mathbb{R}^-, X)$ by*

$$\tilde{\varphi}(r) = \begin{cases} \varphi(t+r) - \varphi(0), & r \leq -t \\ 0, & -t \leq r \leq 0. \end{cases}$$

Further, if $\varphi \in cl(D(A(0)))$, let $g_\varphi(t) = U(t,0)\varphi - \tilde{\varphi}(t), t \geq 0$. Then we have
(i) $\tilde{\varphi} \in BUC(\mathbb{R}^+; E_v)$;
(ii) $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$ if either v additionally satisfies $(v3^*)$ or v additionally satisfies $(v3)$ and $\varphi \in Cv_0(\mathbb{R}^-; X) = \{\varphi \in C(\mathbb{R}^-; X) \mid \lim_{s \to -\infty} v(s)\varphi(s) = 0\}$;
(iii) if $\varphi \in cl(D(A(0)))$,

$$g_\varphi(t)(r) = \begin{cases} \varphi(0), & r \leq -t \\ u_\varphi(t+r), & -t \leq r \leq 0, \end{cases}$$

for all $r \leq 0 \leq t$.

The proof of following proposition is based on the above lemma.

Proposition 4.13. *For $\varphi \in cl(D(A(0)))$, the followings are satisfied,*
(i) $u_\varphi|_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ is bounded if and only if $U(.,s)\varphi : \mathbb{R}^+ \to E_v$ is bounded;
(ii) $u_\varphi|_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ is uniformly continuous if and only if $U(.,s)\varphi : \mathbb{R}^+ \to E_v$ is uniformly continuous.

The results obtained in [61, Theorem 2.4, and Theorem 2.6] can now be stated for the non–autonomous (FDE).

Theorem 4.14. *Given $\varphi \in cl(D(A(0)))$, the following assertions hold:*
(i) If $\{U(t,0)\varphi \mid t \geq 0\}$ is relatively compact in E_v, so is $u_\varphi(\mathbb{R}^+)$ in X. Conversely if v additionally satisfies $(v3)$, $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$, and $u_\varphi|_{\mathbb{R}^+}$ is uniformly continuous, then relative compactness of $u_\varphi(\mathbb{R}^+)$ in X implies that $\{U(t,0)\varphi \mid t \geq 0\}$ is relatively compact in E_v.
(ii) If $U(.,0)\varphi \in AAP(\mathbb{R}^+, E_v)$, then $u_\varphi|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$. Conversely if v additionally satisfies $(v3)$ and $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$, then $u_\varphi|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$ implies $U(.,0)\varphi \in AAP(\mathbb{R}^+, E_v)$.

Theorem 4.15. *Given $\varphi \in cl(D(A(0)))$, the following assertions hold:*
(i) If $\{U(t,0)\varphi \,|\, t \geq 0\}$ is weakly relatively compact in E_v, so is $u_\varphi(\mathbb{R}^+)$ in X. Conversely if v additionally satisfies $(v3)$, $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$, and $u_\varphi|_{\mathbb{R}^+}$ is uniformly continuous, then weak relative compactness of $u_\varphi(\mathbb{R}^+)$ in X implies that $\{U(t,0)\varphi \,|\, t \geq 0\}$ is weakly relatively compact in E_v.
(ii) If $U(.,0)\varphi \in W(\mathbb{R}^+, E_v)$, then $u_\varphi|_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$. Conversely if v additionally satisfies $(v3)$ and $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$, then $u_\varphi|_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$ implies $(U(.,0)\varphi)(0) \in W(\mathbb{R}^+, E_v)$.

Proof of Theorem 4.14 and Theorem 4.15. Using Lemma 4.12, the proof of the above theorems is analogous to the proof of [61, Theorem 2.4, and Theorem 2.6]

4.4 Solutions to (FDE) with relatively compact range

It is known that, in the context of the autonomous (FDE), if the resolvents J_λ^B, $\lambda > 0$ of B are compact and $M \leq \alpha$, then bounded solutions have relatively compact range, [54, Theorem 3.1]. In this section, we discuss some cases, for which the solutions to (FDE) have relatively compact range. In particular we extend the above result to the non-autonomous case: if for some $r_0 > 0$, $B(r_0)$ is α–m–accretive, and the resolvents of $B(r_0)$ are compact, then bounded and uniformly continuous solutions to (FDE) have relatively compact range.
The following lemma will be used later.

Lemma 4.16. *Let $(B(t))_{t\geq 0}$ be a family of operators satisfying (2.1), and u be a mild solution to $\dot{u}(t) + B(t)u(t) \ni g(t)$, $t \in [0,T]$ with $g \in L^1(0,T : X)$, and $T > 0$. Then for $y \in D(J_\lambda^{B(r_0)})$, $r_0 \in [0,T]$, $\lambda > 0$ with $\lambda\alpha < 1$, and for all $0 \leq t_1 \leq t_2 \leq T$, the following inequality holds:*

$$\begin{aligned} \left\| J_\lambda^{B(r_0)} y - y \right\| &\leq \frac{\lambda}{(1-\lambda\alpha)(t_2-t_1)} \|u(t_2) - u(t_1)\| \\ &+ \frac{(2-\lambda\alpha)}{(1-\lambda\alpha)(t_2-t_1)} \int_{t_1}^{t_2} \|u(\tau) - y\| d\tau \\ &+ \frac{C\lambda}{(1-\lambda\omega)(t_2-t_1)} \int_{t_1}^{t_2} \|f(\tau) - f(r_0)\| d\tau \qquad (4.14) \\ &+ \frac{\lambda}{(1-\lambda\omega)(t_2-t_1)} \int_{t_1}^{t_2} \|g(\tau)\| d\tau, \end{aligned}$$

with $C = \max\{L(\left\| J_\lambda^{B(r_0)} y \right\|), L(\sup_{0\leq\tau\leq T} \|u(\tau)\|)\}$.(For the autonomous counterpart, see [3, Lemma 14.7], also compare [47, lemma 5.2]).

Proof. Let $0 \leq t_1 \leq t_2 \leq T$. Since u is a mild solution to $\dot{u}(t) + B(t)u(t) \ni g(t)$ for $0 \leq t \leq T$, we shall apply the integral inequality (2.28) with $r = r_0$, and $[J_\lambda^{B(r_0)}y, B_\lambda(r_0)y] \in B(r_0)$ to obtain

$$\begin{aligned} &\left\|u(t_2) - J_\lambda^{B(r_0)}y\right\| - \left\|u(t_1) - J_\lambda^{B(r_0)}y\right\| \\ \leq \ & \int_{t_1}^{t_2} < g(\tau) - B_\lambda(r_0)y, u(\tau) - J_\lambda^{B(r_0)}y >_+ d\tau \qquad (4.15) \\ + \ & \alpha \int_{t_1}^{t_2} \left\|u(\tau) - J_\lambda^{B(r_0)}y\right\| d\tau + C\int_{t_1}^{t_2} \|f(\tau) - f(r_0)\| d\tau, \end{aligned}$$

where $C = \max\{L\big(\left\|J_\lambda^{B(r_0)}y\right\|\big), L\big(\sup_{s\leq\tau\leq T}\|u(\tau)\|\big)\}$.
We recall that for $a, b \in X$, and $\lambda > 0$

$$< b, a >_+ \leq \frac{\|a + \lambda b\| - \|a\|}{\lambda}.$$

Applying this property to (4.15), and noting that $J_\lambda^{B(r_0)}y + \lambda B_\lambda(r_0)y = y$ we obtain

$$\begin{aligned} \left\|u(t_2) - J_\lambda^{B(r_0)}y\right\| - \left\|u(t_1) - J_\lambda^{B(r_0)}y\right\| \ &\leq \\ \lambda^{-1}\int_{t_1}^{t_2} \|u(\tau) - y + \lambda g(\tau)\| d\tau - \lambda^{-1}\int_{t_1}^{t_2} \left\|u(\tau) - J_\lambda^{B(r_0)}y\right\| d\tau \ &+ \\ \alpha\int_{t_1}^{t_2} \left\|u(\tau) - J_\lambda^{B(r_0)}y\right\| d\tau + C\int_{t_1}^{t_2} \|f(\tau) - f(r_0)\| d\tau. & \end{aligned}$$

Multiplying the above inequality by λ, and using that

$$-\|u(t_1) - u(t_2)\| \leq \left\|u(t_2) - J_\lambda^{B(r_0)}y\right\| - \left\|u(t_1) - J_\lambda^{B(r_0)}y\right\|,$$

we conclude that

$$\begin{aligned} -\lambda\|u(t_1) - u(t_2)\| \leq & \int_{t_1}^{t_2} \|u(\tau) - y\| d\tau \ + \ \lambda\int_{t_1}^{t_2} \|g(\tau)\| d\tau \\ -(1 - \lambda\alpha)& \int_{t_1}^{t_2} \left\|u(\tau) - J_\lambda^{B(r_0)}y\right\| d\tau \ + \ \lambda C\int_{t_1}^{t_2} \|f(\tau) - f(r_0)\| d\tau. \qquad (4.16) \end{aligned}$$

But $\lambda\alpha < 1$, and

$$-\left\|u(\tau) - J_\lambda^{B(r_0)}y\right\| \leq \|u(\tau) - y\| - \left\|y - J_\lambda^{B(r_0)}y\right\|,$$

for all $t_1 \leq \tau \leq t_2$. So inequality (4.16) implies that

$$\begin{aligned} -\lambda\|u(t_1) - u(t_2)\| \leq \int_{t_1}^{t_2} \|u(\tau) - y\| d\tau - (1 - \lambda\alpha)(t_2 - t_1)\left\|y - J_\lambda^{B(r_0)}y\right\| \\ +(1 - \lambda\alpha)\int_{t_1}^{t_2} \|u(\tau) - y\| + \lambda\int_{t_1}^{t_2} \|g(\tau)\| d\tau + \lambda C\int_{t_1}^{t_2} \|f(\tau) - f(r_0)\| d\tau, \end{aligned}$$

which is equivalent to (4.14).

Corollary 4.17. *Under the assumptions of Lemma 4.16, let there exist $r_0 > 0$ such that $B(r_0)$ is α–m–accretive. If u is a bounded solution to*

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni g(t), & t \geq 0 \\ u_0 = x \in cl(D(B(0))), \end{cases} \tag{4.17}$$

with $g \in L^1_{loc}(\mathbb{R}^+; X)$, then for all $t \geq 0$, and $\lambda > 0$ sufficiently small

$$\left\| J_\lambda^{B(r_0)} u(t) - u(t) \right\| \leq \frac{1}{(1-\lambda\alpha)} \|u(t+\lambda) - u(t)\| + \frac{(2-\lambda\alpha)}{\lambda(1-\lambda\alpha)} \int_t^{t+\lambda} \|u(\tau) - u(t)\| d\tau$$

$$+ \frac{1}{(1-\lambda\omega)} \int_t^{t+\lambda} \|g(\tau)\| d\tau + \frac{C_t}{(1-\lambda\omega)} \int_t^{t+\lambda} \|f(\tau) - f(r_0)\| d\tau,$$

where $C_t = \max\{L\big(\left\|J_\lambda^{B(r_0)} u(t)\right\|\big), L\big(\sup_{0 \leq \tau} \|u(\tau)\|\big)\}$.

Proof. Since $B(r_0)$ is α–m–accretive, $J_\lambda^{B(r_0)} u(t)$ is defined for any $t \in \mathbb{R}^+$ and $\lambda > 0$ small enough. Take $t \in \mathbb{R}^+$. Applying Lemma 4.16 with $y = u(t)$, $t_1 = t$, , and $t_2 = t + \lambda$, $\lambda > 0$ with $\lambda\alpha < 1$ implies the desired inequality.

Theorem 4.18. *Let $(B(t))_{t \geq 0} \subset X \times X$ be a family of operators satisfying (B1) with f bounded on $\mathbb{R}^+$. If there exists $r_0 > 0$ such that $B(r_0)$ is α–m–accretive, and $J_\lambda^{B(r_0)}$, $\lambda > 0$ is compact (respectively weakly compact), then bounded and uniformly continuous mild solutions to* (4.17) *with $g \in C_b(\mathbb{R}^+; X)$ (or $g \in L^1(\mathbb{R}^+; X)$), have relatively compact (respectively weakly relatively compact) range.*

Proof. Since u is bounded, and $J_\lambda^{B(r_0)}$ is (weakly) compact, it follows that $\{J_\lambda^{B(r_0)} u(t) \,|\, t \geq 0\}$ is a (weakly) relatively compact set in X. As u is uniformly continuous, from Corollary 4.17, we conclude that, given $\varepsilon > 0$, the orbit $\{u(t) \,|\, t \in \mathbb{R}^+\}$ is ε–close in norm to $\{J_\lambda^{B(r_0)} u(t) \,|\, t \geq 0\}$, and therefore is a (weakly) relatively compact set in X.

Applying the above result to (FDE) we have the following.

Corollary 4.19. *Let u_φ be a bounded and uniformly continuous solution of*

$$\begin{cases} \dot{u}(t) + B(t)u(t) \ni F(t, u_t), & t \geq 0 \\ u_0 = \varphi \in cl(D(A(0))), \end{cases} \tag{4.18}$$

under the conditions of Theorem 2.2 or Theorem 3.9, with $s = 0$, and the control functions f, g bounded on $\mathbb{R}^+$. Moreover, assume that there exists $C' > 0$, such that $\left\|F(t, (u_\varphi)_t)\right\| \leq C'$ for all $t \in \mathbb{R}^+$. If there exists $r_0 > 0$ such that $B(r_0)$ is α–m–accretive and the resolvents $J_\lambda^{B(r_0)}$, $\lambda > 0$, $\lambda\alpha < 1$ are (weakly) compact , then $u_\varphi(\mathbb{R}^+)$ is (weakly) relatively compact .

Definition 4.20. *An evolution operator $U(t,s) : D(s) \to D(t)$ is said to be T-periodic for $T > 0$ if, $U(T+t, T+s) = U(t,s)$, $0 \le s \le t$.*

As we will see, in case that the family $B(t)$ generates a $T-$periodic evolution operator, solutions to (FDE) with relatively compact range are uniformly continuous.

Lemma 4.21. *Let $B(t) \subset X \times X$ be a family of operators satisfying (B1), and such that $(B(t))_{t\ge 0}$ generates a T–periodic evolution operator $U_B(t,0) : cl(D(B(0))) \to cl(D(B(t)))$, $t \ge 0$, $T > 0$. If u is a mild solution to (4.17), then for all $t, s \ge T$, $0 < t - s \le T$;*

$$\|u(t) - u(s)\| \le \int_{kT+r_2}^{kT+r_1} e^{\alpha(kT+r_1-\tau)} \|g(\tau)\| d\tau + \|U_B(r_1, r_2)u(kT + r_2) - u(kT + r_2)\|,$$

where $T \le r_2 < r_1 \le 3T$, and $k \in \mathbb{N}_0$.

Proof. Let $s, t \ge T$ such that $0 < t - s \le T$, and let $U_B(t, 0)$ be the evolution operator generated by $B(t)$. Then

$$\begin{aligned} \|u(t) - u(s)\| &= \|u(kT + r_1) - u(kT + r_2)\| \\ &\le \|u(kT + r_1) - U_B(r_1, r_2)u(kT + r_2)\| \\ &+ \|U_B(r_1, r_2)u(kT + r_2) - u(kT + r_2)\| \end{aligned} \tag{4.19}$$

where $T \le r_2 < r_1 \le 3T$, and $k \in \mathbb{N}_0$.
We observe that $U_B(r_1, r_2)u(kT + r_2)$ is the mild solution of the problem

$$\begin{cases} \dot{v}(\tau) + B(\tau)v(\tau) \ni 0, & kT + r_2 \le \tau \\ v(kT + r_2) = u(kT + r_2). \end{cases}$$

at $kT + r_1$. Therefore (4.19) and (2.29) imply the desired inequality.

Proposition 4.22. *Let u_φ be a solution to (4.18), and the family $(B(t))_{t\ge 0}$ generates a $T-$periodic evolution operator of type α. Also assume that there exists $C' > 0$, such that $\left\|F(t, (u_\varphi)_t)\right\| \le C'$ for all $t \in \mathbb{R}^+$. If $u_\varphi(\mathbb{R}^+)$ is relatively compact, then $u_\varphi : \mathbb{R}^+ \to X$ is uniformly continuous.*

Proof . Let $s, t \ge T$ such that $0 < t - s \le T$, then by Lemma 4.21,

$$\begin{aligned} \|u_\varphi(t) - u_\varphi(s)\| &\le \int_{kT+r_2}^{kT+r_1} e^{\alpha(kT+r_1-\tau)} \|F(\tau, (u)_\tau)\| d\tau \\ &+ \|U_B(r_1, r_2)u(kT + r_2) - u(kT + r_2)\|, \end{aligned} \tag{4.20}$$

where $T \leq r_2 < r_1 \leq 3T$, and $k \in \mathbb{N}_0$. Let $\varepsilon > 0$. We note that for any $r_2 \in [0, 3T]$, and $x \in cl(D(B(r_2)))$, $U_B(., r_2)x$ is uniformly continuous on $[r_2, 3T]$. Since $u_\varphi(\mathbb{R}^+)$ is relatively compact, it follows that for any $r_2 \in [0, 3T]$, the family $\{U_B(., r_2)x \,|\, x \in cl(D(B(r_2))) \cap u_\varphi(\mathbb{R}^+)\}$ is uniformly equicontinuous on $[r_2, 3T]$. In particular, for any $s \in [0, 3T]$, there exists δ_s such that for all $r, r' \in [T, 3T]$, $r, r' > s$ with $|r' - r| < \delta_s$, and for any $k \in \mathbb{N}_0$;

$$\|U_B(r, s)u_\varphi(kT + s) - U_B(r', s)u_\varphi(kT + s)\| < \varepsilon. \tag{4.21}$$

Since $[T, 3T]$ is compact, there exists a finite family $\{s_i\}_{i=1}^n \subset [0, 3T]$ such that $[T, 3T] \subset \cup_{i=1}^n (s_i, s_i + \delta_i/2)$ with $\delta_i = \delta_{s_i}$. Set $\delta = \min\{\delta_i/2\}_{i=1}^n$. Take $r_1, r_2 \in [T, 3T]$ such that $0 < r_1 - r_2 < \delta$. Then there exists $1 \leq j \leq n$ such that $r_2 \in (s_j, s_j + \delta_j/2)$. Moreover, for any $k \in \mathbb{N}_0$,

$$\begin{aligned} \|U_B(r_1, r_2)u_\varphi(kT + r_2) - u_\varphi(kT + r_2)\| &\leq \\ \|U_B(r_1, r_2)u_\varphi(kT + r_2) - U_B(r_1, s_j)u_\varphi(kT + s_j)\| &+ \\ \|U_B(r_1, s_j)u_\varphi(kT + s_j) - U_B(r_2, s_j)u_\varphi(kT + s_j)\| &+ \\ \|U_B(r_2, s_j)u_\varphi(kT + s_j) - u_\varphi(kT + r_2)\| & \end{aligned}$$

Thus (4.21), and (2.29) together with the fact that U_B is an evolution operator of type α imply that

$$\begin{gathered} \|U_B(r_1, r_2)u_\varphi(kT + r_2) - u_\varphi(kT + r_2)\| \leq \\ (e^{\alpha(r_1 - r_2)} + 1) \int_{kT + s_j}^{kT + r_2} e^{\alpha(kT + r_2 - \tau)} \|F(\tau, (u)_\tau)\| d\tau + \varepsilon. \end{gathered}$$

Since F is bounded, from (4.20), it follows that u_φ is uniformly continuous.

Theorem 4.23. *Let $B(t) \subset X \times X$ be a family of operators satisfying (B1), and such that $(B(t))_{t \geq 0}$ generates an evolution operator $U_B(t, 0) : cl(D(B(0))) \to cl(D(B(t)))$, $t \geq 0$. Assume, moreover, that $U_B(t, 0)$ is equicontinuous, and T–periodic for some $T > 0$. If there exists $r_0 > 0$ such that $B(r_0)$ is α–m–accretive, and $J_\lambda^{B(r_0)}$, $\lambda > 0$, are compact (respectively weakly compact), then for all bounded mild solutions u to the Cauchy problem (4.17) with $g \in C_b(\mathbb{R}^+; X)$ (or $g \in L^1(\mathbb{R}^+; X)$), the following assertions hold:*

(i) u is uniformly continuous,

(ii) u has relatively compact (respectively weakly relatively compact) range.

Proof. Let U_B, the evolution operator generated by $B(t)$, be T–periodic and equicontinuous. Let $s, t \geq T$ such that $0 \leq t - s \leq T$. Then by Lemma 4.21,

$$\|u(t) - u(s)\| \leq \int_{kT+r_2}^{kT+r_1} e^{\alpha(kT+r_1-\tau)}\|g(\tau)\|d\tau + \|U_B(r_1, r_2)u(kT + r_2) - u(kT + r_2)\|,$$

where $T \leq r_2 \leq r_1 \leq 3T$, and $k \in \mathbb{N}_0$. Set $K_r = u(\mathbb{R}^+) \cap cl(D(B(r)))$. Since U_B is equicontinuous, then for any $r_2 \in [0, 3T]$, the family of functions $\{U_B(., r_2)x \,|\, x \in K_{r_2}\}$ is uniformly equicontinuous on $[T, 3T]$. Then, from the proof of Proposition 4.22 we read that the last term in the above inequality tends to 0 as $t - s \to 0$. Boundedness of g, then implies that u is uniformly continuous. Assertion (ii) now follows from Theorem 4.18.

Corollary 4.24. *Let $B(t) \subset X \times X$ be a family of operators satisfying (B1), and such that $(B(t))_{t\geq 0}$ generates an evolution operator $U_B(t, 0) : cl(D(B(0))) \to cl(D(B(t)))$, $t \geq 0$. Assume that $U_B(t, 0)$ is equicontinuous, and T–periodic for some $T > 0$. Moreover, suppose that there exists $r_0 > 0$ such that $B(r_0)$ is α–m–accretive, and $J_\lambda^{B(r_0)}$, $\lambda > 0$, $\lambda\alpha < 1$ are compact (respectively weakly compact). If u_φ is a bounded solution to (FDE) such that $\left\|F(t, (u_\varphi)_t)\right\| \leq C$ for all $t \in \mathbb{R}^+$, and some $C > 0$, then u_φ is uniformly continuous and has relatively compact (respectively weakly relatively compact) range.*

Corollary 4.25. *Under the conditions of Corollary 4.24, let u_φ be the mild solution to (FDE) as in Theorem 2.2 or Theorem 3.9 with $\alpha + M \leq 0$. If $U_A(t, 0)$ is also T–periodic, then the following hold:*
(i) If $E = C([-R, 0]; X)$, then $U_A(., 0)\varphi \in AAP(\mathbb{R}^+, C([-R, 0]; X))$.
(ii) In the initial history space $E = E_v$ with v satisfying $(v1)$ and $(v2)$, if v additionally satisfies $(v3)$ and $\tilde{\varphi} \in C_0(\mathbb{R}^+, E_v)$, then $U_A(., 0)\varphi \in AAP(\mathbb{R}^+, E_v)$.

The proof of the above corollary is based on the Theorem 4.9, Theorem 4.14 and the following result proved by A. Haraux.

Theorem 4.26. [27, Theorem 1.1]. *Let U be a periodic contractive evolution operator on a complete metric space (X, d). If $U(\mathbb{R}^+, 0)x$ is relatively compact for $x \in X$, then $U(., 0)x$ is asymptotically almost periodic.*

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